General equilibrium with transactions costs and multiple means of payment*

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December 1999

Abstract

We study general equilibrium in an economy with multiple means of payment characterized by different transactions costs and domains of circulation. We show that if transactions via universally accepted public money are costly, in equilibrium some agents use a 'quasi-money' that has limited domain of circulation but lower transactions costs. We study efficiency and uniqueness of general equilibrium in such an economy and characterize monetary and quasi-monetary prices. We show that such an equilibrium is efficient but different from the Arrow-Debreu one. We also study mutual trade credit as an alternative payment system. In the economy with trade credit there are multiple equilibria that are more efficient than those without trade credit but still not as efficient as Arrow-Debreu equilibrium, too.

*This paper is a technical companion of our paper on Russian economy Guriev, Pospelov and Shaposhnik (forthcoming in Russian). The authors are grateful to Kenneth Arrow, Abhijit Banerjee, Barry Ickes, Jan Lundin, Alexander Petrov, Victor Polterovitch, Alexander Shananin, Andreas Worgotter and other colleagues from the Computing Center and CEMI, Russian Academy of Science, the Central Bank of Russia, participants of RECEP conference, European Workshop on General Equilibrium, seminars at Davis Center at Harvard, MIREA and CEMI.

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1 Introduction

In this paper, we study general equilibrium in an economy with transaction costs and multiple means of payments. A means of payment is characterized by costs of transaction via this means and the domain of circulation within which this means is accepted. Means of payments with lower transactions costs and larger domain of circulation should crowd out the ones with higher costs and smaller domains. It is not obvious, however, whether means of payments with low costs and narrow domains can or cannot co-exist with ones characterized by high costs and broad domains.

In most modern economies, public fiat money has crowded out other means of payment and has emerged as a universal medium of exchange. Fiat money is less costly than cows or gold; it is portable, storable and perfectly divisible. However, the economic history suggests that if public money is characterized by high transactions costs (due to whatever reason), a group of agents may come up with a private money that has lower costs for transactions within the group (and prohibitively high costs for outsiders). A classical example is payment arrangements within a medieval merchant guild. In the Middle Ages, the probability of being robbed was quite high, so that travelling with gold (the universal means of payment) was dangerous. Merchants preferred to pay each other in IOUs that further circulated for transactions among the merchants and then were cleared against each other at the annual fair. The IOUs could not be redeemed for gold by outsiders and therefore had low value to robbers.

A more recent example that has motivated our work is the demonetization experience of the Russian economy in transition. In modern Russia, quasi-money has virtually crowded out the public money in the transactions between firms. According to various sources (Aukuzionek (1998), Guriev and Ickes (1999), Karpov (1997)) firms receive only 20 to 50 per cent of their revenues in cash, while the rest comes in barter, firms' and banks' IOUs, bills of exchange, promissory notes or is simply not paid at all. On the other hand, public money prevails in most consumer transactions and retail markets. It seems to be a common knowledge that several means of payments co-exist in Russian economy.

An important feature of such an economy is a possible multiplicity of relative prices. Surveys of firms involved in barter and quasi-money transactions in Russia show that relative prices in these transactions differ from those in the monetary markets (Karpov (1997), Marin and Schnitzer (1998)). In this paper, we will try to develop a general equilibrium model in which both monetary and quasi-monetary prices will be determined by supply and demand forces in the cash market and quasi-money market, respectively. There are two currencies: cash which is universally accepted but is characterized by high transaction costs and the quasi-money which is only used in the inter-firm transactions but has low transaction costs. Like some authors (Hendley et al. (1998)), we do believe that this counter-intuitive structure of transaction costs is due to underdevelopment of market institutions, lack of rule of law and insecure property rights. Rent-seekers such as organized crime and corrupt bureaucrats prey for cash while the second currency is useless to them. It is easy to take cash offshore or use it to pay personal consumer expenditures. It is hard, however, to use for this purpose firms' IOUs or intermediate goods (such as steel or cement) that these IOUs are redeemable for.[1]

A model with two currencies and different relative prices in cash and quasi-money transactions may be very complex. To keep the model tractable, we study a static deterministic model of general equilibrium in an Arrow-Debreu economy. We assume perfect competition and introduce transaction costs exogenously. We neglect well-known shortcomings of non-monetary exchange, such as search, storage and transportation costs. Still, the model remains rather complex. We show that even if the second currency is free of transaction costs, the social optimum is still not achieved. The equilibrium reaches production possibility frontier but at a 'wrong' point. We also study a model with mutual trade credit as an alternative means of payment. It turns out that general equilibrium with trade credit is very different from one with the quasi-money. The economy is always under the production possibility frontier. Also, there is real indeterminacy of equilibria.

The paper follows research on general equilibrium with transaction costs (Foley (1970), Hahn (1971), Niehans (1971), Heller and Starr (1975)). Our model has two important distinctions. First, we do not consider transaction costs to be physical frictions. In our model the price wedges do not take resources; they are rather similar to taxes imposed on producers by government or private rent-seekers. Along with firms' net profits, these fees are source of income for some consumers and are therefore added up to aggregate demand. Thus it makes sense to compare the equilibrium to the Arrow-Debreu one. The second distinction is that we do allow for bypassing the bid-ask spread via alternative means of payment. This relates our work to the literature on multiple means of payments. Prescott (1987), Lacker and Schreft (1996) and others look at the endogenous choice between cash and credit as a means of payment and analyze implications for welfare costs of inflation and behavior of interest rates.

The other strand of related literature is the modern theory of money and barter. Williamson and Wright (1994), Banerjee and Maskin (1996) and others compare money and barter taking into account search costs and information asymmetries. Usually, these models aim at explaining why money emerges in the economy as a single medium of exchange. Our model though much less sophisticated is to answer the opposite question: why the money still being a universal medium of exchange can *partially* lose its functions and how the economy works

[1]Certainly, in the Arrow-Debreu world, one can try sell the IOUs or steel for cash in a 'secondary' market. The latter transaction would however be a cash transaction and therefore subject to another round of expropriation by competing rent-seekers. See Guriev (1999) for a formal model.

without a common means of payment and unit of account.[2]

The structure of the paper is as follows. In the Section 2 we study a model with transactions costs and a single means of payment. In Subsection 2.1 we introduce the benchmark Arrow-Debreu economy and characterize the Walrasian equilibrium (W-equilibrium). In Subsection 2.2 we introduce transaction costs. We characterize general equilibrium in an economy where each firm faces a bid-ask spread in the product market (BA-equilibrium) and show that this equilibrium is not efficient. In Section 3 we introduce quasi-money and study general equilibrium (Q-equilibrium) and its efficiency. In Section 4 we build a model with inter-firm arrears understood as a system of mutual trade credit. It turns out that in this model general equilibria (C-equilibria) are not locally unique. In Section 5 we consider a numerical example which illustrates the existence, efficiency and real indeterminacy results of the previous sections. Section 6 concludes. Appendix contains proofs.

2 General equilibrium with bid-ask spread

2.1 The benchmark economy

In this Subsection, we describe the benchmark economy. There are n goods and N price-taking firms. Firm $\nu = 1, .., N$ is characterized by a production set $Y^\nu \subset \mathbf{R}^n$ and can choose any net output vector $\vec{y}^\nu \in Y^\nu$. We shall use vector signs to denote vectors in the product space $\mathbf{R}^n$ (or the dual price space), Latin subscripts i, j, k to index components of these vectors and Greek letters ν, μ to index firms.

In order to avoid technical problems, we assume that each firm ν can produce only one good i_ν. Therefore, a firm can alternatively be described by a production function $f^\nu : \mathbf{R}^n_+ \to \mathbf{R}^1_+$ that determines maximum net output of the good i_ν given the inputs of other goods. We assume $f^\nu(\cdot)$ to be monotonic, concave, continuous, bounded from above and

$$f^\nu(\vec{u}) \geq 0;\ f^\nu(\vec{0}) = 0;\ \partial f^\nu / \partial u_{i_\nu} = 0. \tag{1}$$

In terms of technology sets Y^ν, the assumption of being able to produce only one good implies that firm ν's net output $\vec{y}^\nu$ vector consists of one non-negative component $y^\nu_{i_\nu}$ and $n-1$ non-positive components.

$$Y^\nu = \left\{ \vec{y}^\nu \in \mathbf{R}^n : y^\nu_i \leq -u_i,\ i \neq i_\nu;\ y^\nu_{i_\nu} \leq f^\nu(\vec{u});\ \vec{u} \in \mathbf{R}^n_+ \right\}$$

Let us denote m^ν the upper bound on firm ν's output: $y^\nu_i \leq 0 \leq y^\nu_{i_\nu} \leq m^\nu$, $i \neq i_\nu$.

[2]In this sense, our paper is closer to Williamson (1998) where claims on private banks emerge as a private money.

The economy is characterized by the aggregate production set:

$$Y = \left\{ \vec{y} \in \mathbf{R}^n : \vec{y} = \sum_\nu \vec{y}^\nu, \ \vec{y}^\nu \in Y^\nu \right\}.$$

Since production functions are bounded from above, there exists an upper bound on aggregate production vector $\vec{m} < \infty$ such that $m_i = \max_{\vec{y} \in Y} \ y_i$.

We assume that economy is productive, i.e. it is possible to produce positive amounts of all goods:

$$Y \cap \text{int } \mathbf{R}^n_+ \neq \emptyset. \tag{2}$$

We also impose indecomposability condition (similar to 'irreducibility' Assumption 6 in McKenzie (1959)): production of a positive amount of one good requires, directly or indirectly, non-trivial inputs of all other goods:

$$\sum_\nu \vec{y}^\nu \in Y \cap \mathbf{R}^n_{++} \neq \emptyset \Rightarrow \forall i \exists \nu : y_i^\nu > 0. \tag{3}$$

Hereinafter $\mathbf{R}^n_{++} = \mathbf{R}^n_+ \backslash \{\vec{0}\}$. We introduce the indecomposability condition as well as the assumption that each firm cannot produce more than one good, in order to rule out zero prices in equilibrium. These are technical conditions that significantly simplify proofs of the existence theorems below.

Proposition 1 *Let $\vec{r} \in \mathbf{R}^n_{++}$ and $\vec{y}^\nu \in$Argmax$_{\vec{y}^\nu \in Y^\nu} \vec{r}\vec{y}^\nu$. Under the assumptions above: (i) the total output measured in prices $\vec{r}$ is positive*: $\sum_\nu \vec{r}\vec{y}^\nu > 0$; *(ii) only positive prices result in non-negative output: $\sum_\nu \vec{y}^\nu \in \mathbf{R}^n_+$ implies $\vec{r} \in$ int $\mathbf{R}^n_+$.*

We shall also introduce vectors of firms' gross sales $\vec{x}^\nu = \{x_1^\nu, x_2^\nu, ..., x_n^\nu\}$ and gross purchases $\vec{v}^\nu = \{v_1^\nu, v_2^\nu, ..., v_n^\nu\}$:

$$\vec{x}^\nu \geq \vec{0}, \ \ \vec{v}^\nu \geq \vec{0}. \tag{4}$$

The net supply of the firm to the market is $\vec{x}^\nu - \vec{v}^\nu$. Since the main focus of our paper will be on firms, we choose the simplest possible characterization of consumers. We will assume that consumers choose an aggregate consumption vector $\vec{c} \in \mathbf{R}^n_+$.

Definition 1 *A quadruple $\langle \vec{c}, \{\vec{x}^\nu\}, \{\vec{v}^\nu\}, \{\vec{y}^\nu\} \rangle$ is said to be a feasible allocation if: (i) $\vec{y}^\nu \in Y^\nu$, $\vec{c}, \vec{x}^\nu, \vec{v}^\nu \in \mathbf{R}^n_+$; (ii) demand does not exceed supply, which, in turn, does not exceed aggregate net output:*

$$\vec{c} \leq \sum_\nu (\vec{x}^\nu - \vec{v}^\nu) \leq \sum_\nu \vec{y}^\nu; \tag{5}$$

Definition 2 *Two allocations $\langle \vec{c}, \{\vec{x}^\nu\}, \{\vec{v}^\nu\}, \{\vec{y}^\nu\} \rangle$ and $\langle \vec{c}', \{\vec{x}^{\nu\prime}\}, \{\vec{v}^{\nu\prime}\}, \{\vec{y}^{\nu\prime}\} \rangle$ are said to be equivalent in real terms if $\vec{c} = \vec{c}'$ and $\vec{y}^\nu = \vec{y}^{\nu\prime}$ for all ν.*

The consumers maximize utility $U : \mathbf{R}^n_+ \to \mathbf{R}^1_+$. We assume that the utility is monotonic, quasi-concave and non-satiable $\partial U \in \mathbf{R}^n_{++}$. Given income I and prices $\vec{p}$, the consumers maximize the utility function subject to the budget constraint:

$$\vec{c} \in \text{Arg} \max_{\vec{c}' \in \mathbf{R}^n_+,\, \vec{p}\vec{c}' \leq I} U(\vec{c}') \tag{6}$$

There are no initial endowments. Consumers' income comes from producers' net sales revenues so that Walras' law is as follows:

$$I = \vec{p} \sum_\nu (\vec{x}^\nu - \vec{v}^\nu)\,. \tag{7}$$

Definition 3 *A feasible allocation* $\langle \vec{c}, \{\vec{x}^\nu\}, \{\vec{v}^\nu\}, \{\vec{y}^\nu\}\rangle$ *is said to be efficient if there is no such aggregate output vector* $\vec{y}' \in Y$ *that* $\vec{y}' \geq \sum_\nu \vec{y}^\nu$ *and* $\vec{y}\,' \neq \sum_\nu \vec{y}^\nu$.

In an efficient allocation, total output $\vec{y} = \sum_\nu \vec{y}^\nu$ belongs to production possibility frontier ∂Y (PPF), i.e. Pareto frontier of the aggregate production set Y:

$$\partial Y = \{\vec{y} \in \mathbf{R}^n : \vec{y} \in \text{Arg} \max_{\vec{y} \in Y} \vec{r}\,\vec{y} \text{ for some } \vec{r} \in \mathbf{R}^n_{++}\}.$$

In the definition of a feasible allocation, we do not describe how consumers choose consumption, firms choose output and how the prices are determined. In what follows, we shall always assume that consumers are rational in the sense of (6) and the Walras' law holds (7). We will however make different assumptions about firms' objective functions and compare the resulting allocations. We require that net sales do not exceed net output for the whole economy rather than for each firm, because we will allow for alternative channels of exchange below.

Definition 4 *A sextuple* $\langle \vec{p}_W, \vec{c}_W, I_W, \{\vec{x}^\nu_W\}, \{\vec{v}^\nu_W\}, \{\vec{y}^\nu_W\}\rangle$ *is said to be a Walrasian equilibrium (W-equilibrium), if (i)* $\langle \vec{c}_W, \{\vec{x}^\nu_W\}, \{\vec{v}^\nu_W\}, \{\vec{y}^\nu_W\}\rangle$ *is a feasible allocation,* $\vec{p}_W \in \mathbf{R}^n_+$ *and* $I_W \geq 0$; *(ii) consumers are rational (6), (iii) Walras' law holds (7), and (iv) firms maximize their profits*

$$\{\vec{x}^\nu_W, \vec{v}^\nu_W, \vec{y}^\nu_W\} \in Arg \max_{\vec{x}^\nu - \vec{v}^\nu \leq \vec{y}^\nu \in Y^\nu} \vec{p}_W \vec{x}^\nu - \vec{p}_W \vec{v}^\nu\,. \tag{8}$$

According to Definition 4, W-equilibrium is simply the Arrow-Debreu equilibrium in an economy with production. As it has been shown by many authors, under the assumptions above, W-equilibrium exists and is efficient. Moreover, it maximizes consumer utility over the aggregate production set:

$$\vec{c}_W \in \text{Arg} \max_{\vec{c} \in Y} U(\vec{c}).$$

In our model with a single consumer, W-equilibrium is simply the social optimum.

2.2 General equilibrium with bid-ask spread

We will consider an economy in which market participants face a bid-ask spread in product markets i.e. the price that buyers pay is greater than the price that sellers receive. In the medieval time, the price wedge was equal to the expected income of robbers. In Guriev et al. (1999) we discuss in detail a few examples of such a difference that are relevant for the modern Russian economy. The simplest example is sales tax or excise. Another example is rent-seeking in an economy with imperfect property rights. In such an economy, each cash transaction attracts rent-seekers e.g. corrupt bureaucrats and organized crime. Those in control of sales channels create entry barriers for intermediaries and extract rent. This rent is essentially a tax imposed by the rent-seekers on producers. Notice that the bid-ask spread does not violate Walras' law. Both government and rent-seekers contribute to the aggregate demand as consumers.[3]

In this Subsection we suggest a simple framework for general equilibrium analysis of an economy with a bid-ask spread. Under given market prices $\vec{p}$, firms maximize their objective functions that are different from their profits $\vec{p}\vec{x}^\nu - \vec{p}\vec{v}^\nu$. Firms evaluate their sales at prices that are lower than $\vec{p}$. We assume that firm ν maximizes

$$\vec{p}A^\nu \vec{x}^\nu - \vec{p}\vec{v}^\nu, \tag{9}$$

where A^ν are non-negative diagonal matrices. All diagonal elements a_i^ν of matrix A^ν are less or equal than 1:

$$0 < a_i^\nu \leq 1. \tag{10}$$

Thus, $1 - a_i^\nu$ is the bid-ask spread the firm ν faces when it sells the good i. From now on, A^ν are exogenous.

Under the bid-ask spread, buyers (consumers and firms) pay for their purchases at prices $\vec{p}$ while sellers (firms) only get $\vec{p}A^\nu \leq \vec{p}$.

Definition 5 *A sextuple* $\langle \vec{p}_{BA}, \vec{c}_{BA}, I_{BA}, \{\vec{x}_{BA}^\nu\}, \{\vec{v}_{BA}^\nu\}, \{\vec{y}_{BA}^\nu\}\rangle$ *is said to be an equilibrium in the economy with a bid-ask spread (BA-equilibrium), if (i)* $\langle \vec{c}_{BA}, \{\vec{x}_{BA}^\nu\}, \{\vec{v}_{BA}^\nu\}, \{\vec{y}_{BA}^\nu\}\rangle$ *is a feasible allocation, and* $\vec{p}_{BA} \in \mathbf{R}^n_{++}, I_{BA} \geq 0$; *(ii) consumers are rational (6), (iii) Walras law holds (7) and (iv) firms maximize (9):*

$$\{\vec{x}_{BA}^\nu, \vec{v}_{BA}^\nu, \vec{y}_{BA}^\nu\} \in Arg \max_{\vec{x}^\nu - \vec{v}^\nu \leq \vec{y}^\nu \in Y^\nu} \vec{p}_{BA} A^\nu \vec{x}^\nu - \vec{v}^\nu\,. \tag{11}$$

Again, we emphasize that the bid-ask spread does not lead to violation of Walras' law — all $\vec{p}_{BA}\sum_\nu (\vec{x}_{BA}^\nu - \vec{v}_{BA}^\nu)$ is appropriated by consumers. Firms' owners get $\sum_\nu (\vec{p}_{BA}A^\nu\vec{x}_{BA}^\nu - \vec{p}_{BA}\vec{v}_{BA}^\nu)$ while rent-seekers get $\sum_\nu \vec{p}_{BA}(E - A^\nu)\vec{x}_{BA}^\nu$. The transactions costs simply alter firms' objective functions.

[3]The origin of the bid-ask spread is therefore similar to Bisin (1998) where imperfectly competitive financial intermediaries capture rent creating a bid-ask spread in the financial market.

According to the first welfare theorem, W-equilibrium is efficient. If A^ν are close to the unitary matrix E then BA-equilibrium does not differ much from W-equilibrium. However, if A^ν are very small, BA-equilibrium may depart quite far away from W-equilibrium or even become trivial.

Definition 6 *BA-equilibrium is said to be trivial if* $\vec{c}_{BA} = \vec{0}$.

We can derive sufficient conditions for BA-equilibrium to be trivial and sufficient conditions for it to be non-trivial.

Definition 7 *Given production functions* $\{f^\nu\}$, *the set of matrices* $\{A^\nu\}$ *is said to be profitable if*

$$J_- = \min_{\vec{p}\in P^n} \max_{\nu} \max_{0\leq \vec{u}\leq \vec{m}} \left[\vec{p}a^\nu_{i_\nu} f^\nu(\vec{u}) - \vec{p}\vec{u}\right] > 0 \tag{12}$$

and non-profitable if

$$J_+ = \max_{\vec{p}\in P^n} \max_{\nu} \max_{0\leq \vec{u}\leq \vec{m}} \left[\vec{p}a^\nu_{i_\nu} f^\nu(\vec{u}) - \vec{p}\vec{u}\right] \leq 0$$

Hereinafter $P^n = \{\vec{p} : p_i \geq 0, \sum_{i=1}^n p_i = 1\}$ *is the price simplex.*

Theorem 1 *Under the assumptions above, BA-equilibrium exists. BA-equilibrium is trivial whenever matrices* $\{A^\nu\}$ *are non-profitable, and is not trivial whenever the matrices* $\{A^\nu\}$ *are profitable.*

The set of profitable matrices $\{A^\nu\}$ is not empty. It is open and always contains the unit matrices $\{E\}$ (since the economy is productive (2)). The set of non-profitable matrices is not empty as well, for it contains trivial matrices $\{0\}$. The Theorem does not cover the intermediate case of matrices which are neither profitable nor non-profitable. For these matrices, BA-equilibrium exists but it is hard to figure out whether it is trivial or not.

Walrasian equilibrium is the limiting case of BA-equilibrium at $A^\nu = E$. An important difference between BA- and W-equilibria is that BA-equilibrium is inefficient in the generic case even if it is not trivial.

Proposition 2 *In the BA-equilibrium firms' net output vectors* $\vec{y}^\nu_{BA}$ *belong to Pareto frontiers of firms' production sets:* $y^\nu_{BA\,i_\nu} = f^\nu(-\vec{y}^\nu_{BA})$. *Assume that: (i) the production functions* f^ν *are differentiable in the points* $-\vec{y}^\nu_{BA}$*; (ii) there are non-trivial transactions costs* $A^\nu < E$ *at least for some* ν;[4] *(iii) at least one firm has non-trivial inputs:* $p^{\nu\,i}_{BA} y^\nu_{BA\,i} < 0$ *for some* i, ν. *Then BA-equilibrium is inefficient.*

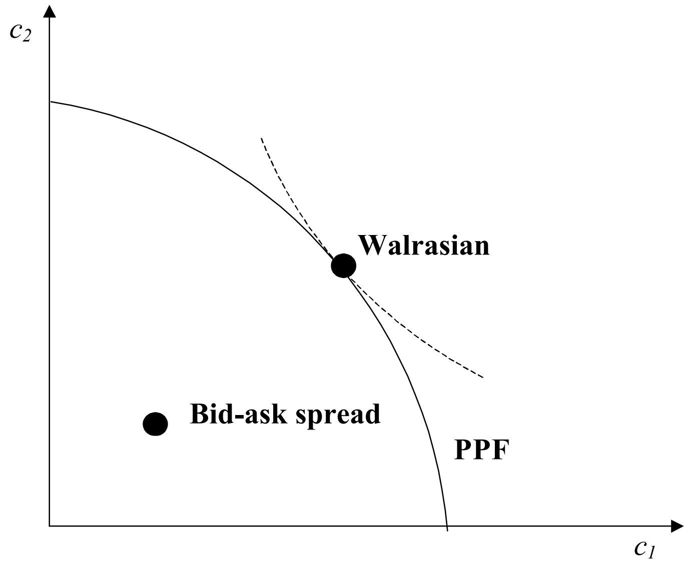

Figure 1: The equilibrium in an economy with a bid-ask spread (BA-equilibrium) is inside the aggregate production set Y. The dotted line is an indifference curve of the consumer utility function.

Figure 1 illustrates Theorem 2 for an economy with two goods. Walrasian equilibrium is the point where consumer indifference curve is tangent to the PPF while the BA-equilibrium is inside the aggregate production set.

If the equilibrium were efficient (even being different from Arrow-Debreu equilibrium) there would exist such relative prices (proportional to the vector that is normal to PPF at the equilibrium) that the producers as a whole would maximize the net output in these prices. Or, alternatively, we could find such utility functions that the equilibrium would be Pareto optimal. In this sense any allocation on PPF can be described by certain optimality principle. However, BA-equilibrium which lies below PPF cannot be characterized by any reasonable optimality principle (except maybe for γ-efficiency as defined in Dubey and Geanakoplos (1992)).

3 General equilibrium with quasi-money

The model above assumes that the economy has a single means of payment (money) that serves as a universal unit of account. However, it does not have to be so. If it is possible to introduce a quasi-money with lower transactions costs even though it has a limited domain of circulation, the firms may choose to do this.

We will study general equilibrium in a model with two means of payment. One is money which has a universal domain of circulation by definition, i.e. can be used in every transaction. The alternative currency is 'quasi-money' with domain of circulation limited to inter-firm transactions with the second system of prices that emerges in general equilibrium.[5]

The assumption that quasi-money can only be used in the inter-firm transactions is quite an arbitrary one but is roughly appropriate. Households do not usually have access to the quasi-monetary transactions since there are fixed costs of entry that are negligible for firms but very high for individuals.

Each firm can buy and sell in two markets: in the cash market at prices $\vec{p}$ dollars per unit and in the quasi-money market at prices $\vec{q}$ quasi-dollars per unit. We study the case where transaction costs in the quasi-money transactions are lower than those in the cash market (otherwise the quasi-money would be crowded out by cash). For the clarity's sake we will consider the extreme case where the exchanges via quasi-money incur no transaction costs. Briefly comparing the two

[4]Hereinafter the inequalities between diagonal matrices are understood as inequalities between vectors composed of their diagonal elements.

[5]We have in mind firms', banks' and governments' liabilities such as IOUs, promissory notes and bills of exchange. There is a lot of evidence that these 'securities' are indeed used as a medium of exchange in Russian economy changing many hands before being paid back by the issuers. Moreover, many barter transactions are facilitated by such IOUs that are redeemable in kind (Carlin et al. (forthcoming)).

means of payments, one should note that: (a) the quasi-money transactions are not taxed or at least not taxed in full; (b) the rent seekers capture rents in quasi-money transactions as well but since corrupt bureaucrats and racketeers would prefer to collect cash (to take it offshore or to spend it in the consumer good market) rather than quasi-money there is a greater concentration of rent-seekers in the cash markets.

Firms choose output $\vec{y}^{\nu}$, sales $\vec{x}^{\nu}$ and purchases $\vec{v}^{\nu}$ in the cash market and net sales $\vec{t}^{\nu}$ in the quasi-money market in order to maximize monetary profit

$$\vec{p}A^{\nu}\vec{x}^{\nu} - \vec{p}\vec{v}^{\nu} \tag{13}$$

subject to the technology constraints

$$\vec{x}^{\nu} - \vec{v}^{\nu} + \vec{t}^{\nu} \leq \vec{y}^{\nu}, \;\; \vec{x}^{\nu} \geq 0, \;\; \vec{v}^{\nu} \geq 0, \;\; \vec{y}^{\nu} \in Y^{\nu}. \tag{14}$$

The firm must also must satisfy the quasi-money budget constraint

$$\vec{q}\vec{t}^{\nu} \geq 0. \tag{15}$$

In other words, each firm should get enough quasi-dollars for its sales in the inter-firm market in order to pay for its purchases in this market.[6] The firm does not appreciate the quasi-money per se; it is the monetary income (13) that the firm's owners maximize.

Proposition 3 *The maximization problem (13)-(15) has a finite solution if and only if: (a) for all i, j such that quasi-money prices are positive $q_i, q_j > 0$, the inequality $p_i a_i^{\nu}/q_i \leq p_j/q_j$ holds and (b) for all k such that quasi-money price is trivial $q_k = 0$, the cash price is also trivial $p_k = 0$. The solution is as follows:*

(a) The net output $\vec{y}^{\nu}$ maximizes profit in prices $\vec{q}$ i.e. $\vec{y}^{\nu} \in Argmax_{\vec{y} \in Y^{\nu}} \vec{q}\vec{y}$.

(b) The maximum value of the objective function is $\theta^{\nu}\Pi^{\nu}(\vec{q})$ where $\theta^{\nu} = \max_{i:q_i>0} \frac{p_i a_i^{\nu}}{q_i}$ and $\Pi^{\nu}(\vec{q}) = \max_{\vec{y}^{\nu} \in Y^{\nu}} \vec{q}\vec{y}^{\nu}$.

(c) For all i such that $p_i a_i^{\nu}/q_i < \theta^{\nu}$ sales in the cash market are trivial $x_i^{\nu} = 0$. For all j such that $p_j/q_j > \theta^{\nu}$ purchases in the cash market are trivial $v_j^{\nu} = 0$. The other variables are determined by the constraints (15) and (14) for i such that $q_i > 0$ that are binding.

[6]The optimization problem (13)-(15) also describes a firm in an economy with barter exchanges between firms. Suppose that in addition to buying and selling for money, the firm can buy and sell in a barter market with relative prices $\vec{q}$ to which consumers have no access. If we neglect storage costs and problems with double coincidence of wants, the model with barter would be precisely the same as one with quasimoney. Polterovitch (1998) argues that 'institutialization' of barter in Russia has driven these costs down to very low levels.

The firm maximizes output in quasi-monetary prices $\vec{q}$, then chooses the good with the lowest relative bid-ask spread and sells as much of this good as the quasi-monetary budget constraint allows.

Now we need to determine prices. It is reasonable to assume that the quasi-money prices are set by (quasi-) market forces to equalize supply and demand in the inter-firm market:

$$\sum_{\nu} \vec{t}^{\nu} \geq \vec{0}. \tag{16}$$

Definition 8 *An octuple* $\left\langle \vec{p}_Q, \vec{q}_Q, \vec{c}_Q, \{\vec{x}^{\nu}_Q\}, \{\vec{v}^{\nu}_Q\}, \{\vec{t}^{\nu}_Q\}, I_Q \right\rangle$ *is said to be a general equilibrium in the economy with quasi-money (Q-equilibrium) if (i)* $\left\langle \vec{c}_Q, \{\vec{x}^{\nu}_Q\}, \{\vec{v}^{\nu}_Q\}, \{\vec{y}^{\nu}_Q\} \right\rangle$ *is a feasible allocation, and* $\vec{p}_Q \in \mathbf{R}^n_{++}, I_Q \geq 0$; *(ii) consumers are rational (6); (iii) Walras law holds (7); (iv)* $\{\vec{x}^{\nu}_Q, \vec{v}^{\nu}_Q, \vec{t}^{\nu}_Q, \vec{y}^{\nu}_Q\}$ *maximizes (13) subject to (14)-(15) for each* ν *; (v) demand does not exceed supply in the quasi-money market (16).*

Theorem 2 *Under the assumptions above, Q-equilibrium exists and is efficient. In Q-equilibrium, if* $a^{\nu}_i < 1$ *for all* i, ν *then all inputs are purchased through the quasi-money market:* $\vec{v}^{\nu}_Q = 0$ *for all* ν.

The Proof of Theorem 2 heavily relies on the fact that prices are positive in equilibrium and therefore on the indecomposability assumption. The difficulties with dropping this assumption are related to differences in θ^{ν} which are due to different transaction costs for different firms. If all firms faced the same transaction costs $A^{\nu} = A$, the proof would be much simpler. In particular, it turns out that Q-equilibrium would maximize sum of firms' monetary profits over the aggregate production set.

Proposition 4 *Let* $A^{\nu} = A$, $\nu = 1, \ldots, N$ *then there exists a Q-equilibrium with* $\vec{q}_Q = \lambda \vec{p}_Q A$, $\lambda > 0$. *In this equilibrium* $\vec{x}^{\nu}_Q, \vec{v}^{\nu}_Q, \vec{t}^{\nu}_Q, \vec{y}^{\nu}_Q$ *maximize firms' aggregate effective profit* $\sum_{\nu} [\vec{p}_Q A \vec{x}^{\nu} - \vec{p}_Q \vec{v}^{\nu}]$ *subject to technology constraints (14) and the balance constraint in the quasi-monetary market (16).*

Q-equilibrium is locally unique. In order to find monetary and quasi-monetary prices $\vec{p}$ and $\vec{q}$ we have $2n$ inequalities (that are binding unless the corresponding price is zero) $\sum_{\nu}(\vec{x}^{\nu} - \vec{v}^{\nu}) \leq \vec{c}$ and $\sum_{\nu} \vec{t}^{\nu} \geq \vec{0}$. One equation can be excluded since $\sum_{\nu} \vec{q}\vec{t}^{\nu} = 0$. The latter is a linear combination of the constraints (16) which are binding according to Proposition 3. Similarly, we can exclude another equation due to Walras' law. Thus we have $2n - 2$ equations for $2(n-1)$ relative prices.

Let us check whether the general equilibrium in the economy with quasi-money can coincide with the Walrasian equilibrium. For simplicity's sake let us assume again that PPF is smooth at least at the Arrow-Debreu point so that there is only one vector normal to PPF at this point. Then Q-equilibrium coincides with W-equilibrium if and only if $\vec{q}_Q = \vec{p}_Q = \vec{p}_W$. Thus, Q- and W-equilibria coincide only in the case when A is proportional to E (as shown in Section 5). An example of this seemingly non-generic case is an economy with a sales tax on cash transactions. If the firms are able to avoid this tax in the quasi-monetary transactions (e.g. by declaring a negligible cash value of goods sold through barter) the tax only applies to the sales of consumer goods and therefore is equivalent to a consumption tax. If the tax rates are the same for all goods then the tax is not distortive and the Q-equilibrium coincides with the social optimum.

In the generic case the relative prices in the cash market and in the quasi-money market are different. The economy has two means of payment and two units of account. Consumers make their choices using relative prices $\vec{p}$ while firms base their decisions on the prices $\vec{q}$.

4 General equilibrium with mutual credit

Another way to overcome high transactions costs is a system of mutual trade credit. Since cash transfers among firms are costly, firms can create a large clearing system in order to reduce amount of gross inter-firm cash flows. This alternative payment system does not require introduction of a second currency. Rather, it is a system of trade credit linked to the primary money which partially loses the function of means of payment but remains the only unit of account. In this Section we build a general equilibrium model of an economy with a bid-ask spread and mutual trade credit. Again, we assume that there are two markets in which firms can buy and sell goods. First, there is a free market with prices $\vec{p}$, and second, there is an inter-firm market in which prices are lower. In the latter, firms only pay $\vec{p} - \vec{s}$, where $0 \leq \vec{s} \leq \vec{p}$. The remaining $\vec{s}$ dollars per unit are recorded as arrears: an asset in the seller's balance sheet and a liability in the buyer's. Each firm chooses how much to buy and sell in each market taking prices $\vec{p}$ and $\vec{p} - \vec{s}$ as given. The firms may want to sell at lower prices in order to be able to buy at lower prices. Firms accept customers' arrears as a special kind of asset in order to justify their own arrears to suppliers which arise due to the firm's purchases at prices $\vec{p} - \vec{s}$.

The firm chooses output $\vec{y}^\nu$, sales $\vec{x}^\nu$ and purchases $\vec{v}^\nu$ in the free market, sales $\vec{z}^\nu$ and purchases $\vec{w}^\nu$ in the inter-firm market in order to maximize monetary profit

$$J = \vec{p}A^\nu \vec{x}^\nu + (\vec{p} - \vec{s})A^\nu \vec{z}^\nu - \vec{p}\vec{v}^\nu + (\vec{p} - \vec{s})\vec{w}^\nu \tag{17}$$

subject to technology constraints

$$\vec{x}^\nu + \vec{z}^\nu - \vec{v}^\nu - \vec{w}^\nu \leq \vec{y}^\nu, \tag{18}$$

$$\vec{x}^\nu \geq \vec{0}, \vec{z}^\nu \geq \vec{0}, \vec{v}^\nu \geq \vec{0}, \vec{w}^\nu \geq \vec{0}, \ \vec{y}^\nu \in Y^\nu \tag{19}$$

and the solvency constraint

$$\vec{s}\vec{z}^{\nu} \geq \vec{s}\vec{w}^{\nu}. \tag{20}$$

The solvency constraint (20) sets the limit on the amount of trade credit the firm can get from the suppliers: the firm is allowed to borrow no more than others borrow from the firm.[7]

Proposition 5 *The optimization problem above has finite solution(s). The solution(s) is characterized as follows:*

(a) the constraints (18) and (20) are binding

(b) $u^{\nu}_{i_\nu} = 0$; for all $i \neq i_\nu$ holds $x^{\nu}_i = 0$;

(c) if $s_i < p_i$ then $w^{\nu}_{i_\nu} = 0$ and for all $i \neq i_\nu$ holds $z^{\nu}_i = 0$; if $s_i = p_i$ then for all $i \in I^{\nu}_{+}$ at least on one solution holds $w^{\nu}_{i_\nu} = 0$ and $z^{\nu}_i = 0$ for all $i \neq i_\nu$;

(d) if there is at least one i such that $x^{\nu}_i > 0$ then $\vec{w}^{\nu} = \vec{0}$;

(e) if there is at least one i such that $u^{\nu}_i > 0$ then $\vec{x}^{\nu} = \vec{0}$.

The solutions can be classified by the value of Lagrange multiplier λ^{ν} for the constraint (20). This dual variable takes its value to minimize the Lagrange function

$$\begin{aligned} L \;=\; & \max_{\vec{y}^{\nu} \in Y^{\nu}} \vec{r}^{\nu}\vec{y}^{\nu} + \max_{\vec{x}^{\nu} \geq \vec{0}} (\vec{p}A^{\nu} - \vec{r}^{\nu})\vec{x}^{\nu} + \max_{\vec{v}^{\nu} \geq \vec{0}} (-\vec{p} + \vec{r}^{\nu})\vec{v}^{\nu} + \\ & + \max_{\vec{z}^{\nu} \geq \vec{0}} \left((\vec{p} - \vec{s})A^{\nu} - \vec{r}^{\nu} + \lambda^{\nu}\vec{s}\right)\vec{z}^{\nu} + \max_{\vec{w}^{\nu} \geq \vec{0}} \left(-(\vec{p} - \vec{s}) + \vec{r}^{\nu} - \lambda^{\nu}\vec{s}\right)\vec{w}^{\nu}, \end{aligned}$$

where $\vec{r}^{\nu}$ is the Lagrange multiplier for the constraint (18). Using straightforward calculations, we obtain $a^{\nu}_{i_\nu} \leq \lambda^{\nu} \leq 1$. There can be three generic cases.

1. $\lambda^{\nu} = a^{\nu}_{i_\nu}$. In this case the firm sells in the cash market.

2. $\lambda^{\nu} = 1$. In this case the firm buys in the cash market.

3. $a^{\nu}_{i_\nu} < \lambda^{\nu} < 1$. In this case the firm does not operate in the cash market at all $\vec{x}^{\nu} = \vec{v}^{\nu} = 0$.

The firm's supply function cannot be characterized as clearly as in the case of quasi-money/barter economy. Let us compare it to the supply function in our benchmark economies: Arrow-Debreu economy and the economy with a bid-ask spread.

[7] In many countries bankruptcy legislation allows to start bankruptcy proceedings as soon as constraint (20) is violated. Even under the alternative bankruptcy legislation which allows to trigger bankruptcy whenever a single creditor is not paid for a certain period of time, liquidation is unlikely to happen as long as (20) is satisfied.

Proposition 6 *A firm that maximizes (17) subject to (18)-(20) chooses the same output* $\vec{y}^\nu$ *=Argmax*$_{\vec{y}\in Y^\nu}\vec{p}^\nu\vec{y}$ *as a firm in the Walrasian economy if and only if* $\vec{s}=\vec{p}$*. If* $\vec{s}=0$*, then the firm chooses the same output as it would choose in the BA-economy. If* all *firms have BA-supply functions then condition* $\vec{s}=0$ *is also a necessary condition.*

Firms take the rates of underpayments s_i for each good as given. These rates are determined by equilibrium conditions. We believe that prices $\vec{p}$ are to equalize supply and demand in the free market while prices $\vec{p}-\vec{s}$ are determined from the inter-firm market equilibrium conditions:

$$\sum_\nu \vec{z}^\nu - \sum_\nu \vec{w}^\nu \geq \vec{0}. \tag{21}$$

Definition 9 *A combination* $\langle \vec{p}_C, \vec{s}_C, \vec{c}_C, \{\vec{x}^\nu_C\}, \{\vec{v}^\nu_C\}, \{\vec{y}^\nu_C\}, \{\vec{z}^\nu_C\}, \{\vec{w}^\nu_C\}, I_C\rangle$ *is said to be a general equilibrium in the economy with mutual credit (C-equilibrium) if (i)* $\langle \vec{c}_C, \{\vec{x}^\nu_C\}, \{\vec{v}^\nu_C\}, \{\vec{y}^\nu_C\}\rangle$ *is a feasible allocation, and* $\vec{p}_C \in \mathbf{R}^n_{++}, I_C \geq 0, 0 \leq \vec{s}_C \leq \vec{p}_C$*; (ii) consumers are rational (6); (iii) Walras law holds, (7) and (iv)* $\{\vec{x}^\nu_C, \vec{v}^\nu_C, \vec{t}^\nu_C, \vec{y}^\nu_C, \vec{z}^\nu_C, \vec{w}^\nu_C\}$ *maximizes (17) subject to (18)-(20) for each* ν *; (v) the inter-firm market is in equilibrium (21).*

Theorem 3 *Let matrices* $\{A^\nu\}$ *be profitable. Then there is such* $\bar{\chi} > 0$ *that for every* $\chi \in [0, \bar{\chi}]$ *there exists a C-equilibrium with* $\vec{p}_C > \vec{0}$ *and* $\sum_i s^i_C = \chi \sum_i p^i_C$*.*

Unlike the other models above, the model with mutual credit has real indeterminacy of equilibria. There are n inequalities $\vec{c}_C \leq \sum_\nu \vec{x}^\nu_C - \sum_\nu \vec{v}^\nu_C$ one of which can be excluded due to Walras' law. Again, there are n equilibrium conditions (21) one of which can be excluded due to the fact that constraints (20) are binding for all ν. We have $2n$ unknowns $\vec{p}_C, \vec{s}_C$ with one degree of freedom. Indeed, the equilibrium will remain the same in real terms if we multiply both $\vec{p}_C$ and $\vec{s}_C$ by a scalar factor. Thus we have $2n-2$ equations for $2n-1$ unknowns. In the generic case, there should be a one-dimensional continuum of equilibria. Unlike the quasi-money model these equilibria differ not only in $\vec{s}$ but also in real terms. Since there is a nominal anchor $\vec{p}$, change in $\vec{s}$ would, generally speaking, lead to a change in outputs and therefore supply to the free market (see the example in the Section 5).

Among these equilibria there is always an equilibrium with $\vec{s}_C = 0, \vec{p}_C = \vec{p}_{BA}$ which is equivalent in real terms to BA-equilibrium. Indeed, in the case $\vec{s} = 0$ firms' outputs are determined from the same maximization problems as in the BA-economy. This is the equilibrium where firms' debt is worth nothing and firms charge the same prices in the inter-firm market as in the free market. As already mentioned this allocation is not efficient. There may exist other, more efficient equilibria (as in the example below). In some cases the equilibria may even reach the PPF. It turns out that in the latter case the C-equilibrium coincides with the Walrasian one $\vec{s}_C = \vec{p}_C = \vec{p}_W$.

The economy with mutual credit may have only one equilibrium that belongs to the PPF and it must be the Walrasian equilibrium. In this equilibrium the inter-firm market becomes entirely non-monetary. The firms choose whether to supply the product to the free market or exchange it for inputs in the non-payments market. In this case the mutual credit restores efficiency completely whatever A^ν are.

This equilibrium does not have to exist. It is no wonder since the nonpayment equilibrium $\vec{s}_C = \vec{p}_C$ is virtually equivalent to the quasi-money (or barter) one with $\vec{q}_Q = \vec{p}_Q = \vec{p}_W$ if there is any. In the generic case neither of these exists. Also if one of them exists (non-generic case) then the other exists as well.

Proposition 7 *Let the PPF be smooth in the vicinity of W-equilibrium. If C-equilibrium is efficient then $\vec{s}_C = \vec{p}_C = \vec{p}_W$. A C-equilibrium that lies on PPF exists if and only if Q-equilibrium coincides with W-equilibrium. In the latter case C-equilibrium also coincides with W-equilibrium.*

The intuition for real indeterminacy is as follows. The inter-firm market works without the second currency. Rather, it is serviced by mutual trade credit. Unlike the free market, the inter-firm market does not have transactions costs. Therefore the efficiency is the greater the less cash and the more inter-firm credit is used in the inter-firm transactions. Due to the solvency constraint, even in a general equilibrium model we have an externality-like effect. If a firm pays a full price it will not want to be underpaid itself. On the other hand, if the firm is not paid in full it will have a justification for borrowing through underpaying its suppliers. Certainly, this argument is by no means straightforward and we resort to a numerical example to show that this indeed happens.

5 A numerical example

Consider an economy with two firms and two goods. The first firm makes good 1 out of good 2 while the second firm makes 2 out of 1. The production sets are given by production functions: $Y^1 = \{\vec{y} : y_2 \le 0 \le y_1 \le f^1(-y_2)\}$, $Y^2 = \{\vec{y} : y_1 \le 0 \le y_2 \le f^2(-y_1)\}$, where $f^\nu(u) = \sqrt{2\gamma^\nu u}$ with productivity parameters $\gamma^1 \ge \gamma^2$. Also, A^ν are given. For simplicity's sake we set $a_1^2 = a_2^1 = 0$. Denote $\beta^1 = 1/a_1^1$, $\beta^2 = 1/a_2^2$ ($\beta^1, \beta^2 > 1$).

Suppose the consumer utility is linear $U(\vec{c}) = c_1 + c_2$ for $c_1, c_2 \ge 0$. Then in any non-trivial equilibrium both firms will supply to the cash market and the prices for both goods will be equal to each other $p_1 = p_2 = 1/2$.

To find the equilibrium in the economy with quasi-money we assume that both firms maximize profits in prices $\vec{q}$ and then find equilibrium prices q_1/q_2 from the condition (16). Straightforward calculations give $q_1/q_2 = (\gamma^2/\gamma^1)^{1/3}$.

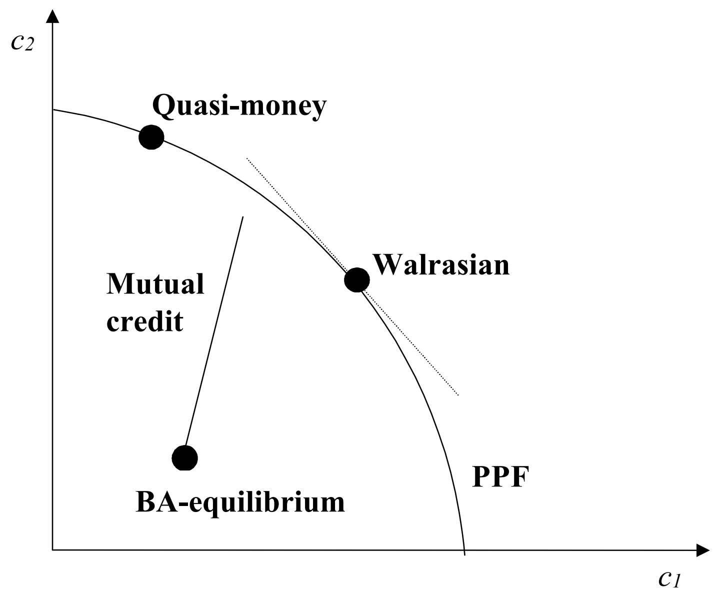

Figure 2: Equilibria in the generic case $\gamma^1 \neq \gamma^2$. In this diagram, $\gamma^1 = 5$, $\gamma^2 = 3$, $\beta^1 = 2$, $\beta^2 = 3$.

The Q-equilibrium always lies on the PPF. It coincides with W-equilibrium only if $\gamma^1 = \gamma^2$ (non-generic case).

In the economy with mutual credit, firms never buy in the cash market. The first firm solves $\max_{w \geq 0} p_1 f^1(w) - (p_2\beta^1 - s_2(\beta^1 - 1))w$ and then sets $w_2^1 = -y_2^1 = 0.5\gamma^1(p_1/(p_2\beta^1 - s_2(\beta^1 - 1)))^2$, $y_1^1 = f^1(-y_2^1)$ and $z_1^1 = s_2 w_2^1/s_1$. The second firm solves a similar problem and sets $w_1^2 = -y_1^2 = 0.5\gamma^2(p_2/(p_1\beta^2 - s_1(\beta^2 - 1)))^2$, $y_2^2 = f^1(-y_1^2)$ and $z_2^2 = s_1 w_1^2/s_2 w_2^1$. The two equilibrium conditions for the underpayment market $z_1^1 = w_1^2$ and $z_2^2 = w_2^1$ are equivalent to each other and to $s_1 w_1^2 = s_2 w_2^1$. Thus we have only one condition for s_1 and s_2. For the particular utility and cost functions selected above it is as follows:

$$\frac{s_1\gamma^2}{(\beta^2 - 2s_1(\beta^2 - 1))^2} = \frac{s_2\gamma^1}{(\beta^1 - 2s_2(\beta^1 - 1))^2} \tag{22}$$

Left-hand side increases with s^1 and right-hand sides increases s^2. Thus, in the square $\{(s_1, s_2) : 0 \leq s_1, s_2 \leq 1/2\}$ the equation (22) defines an increasing curve which connects points $s_1 = s_2 = 0$ (BA-equilibrium) and $s_1 = 1/2$, $s_2 = s_2^* \leq 1/2$. The parameter s_2^* depends upon the ratio of γ^1/γ^2 and equals 1/2 only if $\gamma^1 = \gamma^2$. There is a (one-dimensional) continuum of equilibria. The equilibria differ not only in $\vec{s}$ but also in terms of real variables (outputs depend upon s_1/s_2 which is different in the different equilibria). The continuum always includes BA-equilibrium. In the non-generic case $\gamma^1 = \gamma^2$ it also includes Arrow-Debreu equilibrium. In the consumption space c_1, c_2 the equilibria are represented by a continuous curve that goes from the BA-equilibrium in the direction of the competitive equilibrium (that is reached only if $\gamma^1 = \gamma^2$).

6 Conclusions

We have developed a general equilibrium model of an economy with a bid-ask spread. In such an economy, there may emerge an inter-firm market in which firms buy and sell using either quasi-money, or barter exchanges, or mutual trade credit. We believe that the model can explain the sustainability of Russia's 'virtual economy' in which different payment systems co-exist in a steady state, in some sense violating the Gresham law. In response to the challenge of imperfections of the primary payment system, the economy creates a supplementary one. If cash transactions are costly, the firms issue quasi-money themselves or lend them to each other as long as their liabilities do not exceed what is owed to them. These ideas are quite straightforward and definitely not new. Our contribution is that we put all of these in a consistent general equilibrium framework showing how such an economy works and why one payment system still does not crowd out the other one.

We have also shown that despite the models with quasi-money and mutual credit look similar, the properties of equilibria are very different both in terms of efficiency and uniqueness. There is a continuum of equilibria in the economy with trade credit which suggest that policymakers may be right when suggesting a hysteresis effect in an arrears economy: tightening money supply leads to real changes while money printing would simply result in a proportional increase in prices $\vec{p}$ and $\vec{s}$ and would not have any real impact. The model also explains why barter and quasi-money have been crowding out arrears in the Russian economy: the institution of quasi-money is efficient while that of mutual trade credit is not. (Introduction of the trade credit decreases inefficiency in an economy with transactions costs, but not entirely.)

An important result is that even if we neglect all transaction costs in the quasi-monetary transactions, the economy would get to the PPF, but at a 'wrong' point. Emergence of the quasi-monetary market distorts relative prices and the social optimum is not achieved. Thus even the 'institutionalized' barter with low search and storage costs is costly to the economy.

Our model is rather general and characterizes the equilibrium whatever the origin of the bid-ask spread is. The model applies to any economy where there are two means of payments with different domains of circulation and structure of transactions. For any particular application, though, it may be worth while endogenizing what we have taken as given in an attempt to make the model tractable. First, a natural extension of the model is to endogenize the transactions costs and explicitly describe where the bid-ask spread comes from. This could also bring about endogenous domains of circulation for each means of payment. For example, one can think of an economy where the transportation cost of cash money is proportional to the amount paid while the electronic money takes a fixed cost per transaction no matter how large the transaction is. Therefore cash would dominate smaller transactions while electronic money would be more common in larger transactions. Relative prices in cash dollars and electronic dollars may be different. Another straightforward extension of our analysis is to describe monetary and quasi-monetary balances explicitly. One can introduce Clower cash-in-advance constraints so that non-trivial holdings of cash and quasi-money appear in equilibrium.

Appendix: Proofs

PROOF OF PROPOSITION 1. Take $\vec{y} = \sum_\nu \vec{r}\vec{y}^\nu$. By definition $\vec{y} \in \text{Argmax}_{\vec{y}' \in Y} \vec{r}\vec{y}'$. Thus $\vec{y} \in \partial Y = \{\vec{z} \in Y \colon \forall \vec{z}' \in Y \quad \exists j : z_j \geq z'_j\}$ (∂Y is Pareto frontier of Y). For a productive economy $\vec{0} \notin \partial Y$, therefore $\sum_\nu \vec{r}\vec{y}^\nu > 0$.

Suppose now that the economy is productive, indecomposable and $\sum_\nu \vec{y}^\nu \in \mathbf{R}^n_+$. Let K be the set of goods with zero prices $K = \{i = 1, \ldots, n \mid r_i = 0\}$ and - — the set of firms that produce these goods - $= \{\nu = 1, \ldots, N \mid i_\nu \in K\}$. Since $\vec{y}^\nu \in \text{Argmax}_{\vec{y}^\nu \in Y^\nu} \vec{r}\vec{y}^\nu$, conditions (1) imply $y^\nu_j = 0$ if $j \notin K$, $\nu \in$ - . Denote $\vec{w} = \sum_{\nu \in} \vec{y}^\nu$ and prove that $\vec{w} = \vec{0}$. If $w_j < 0$ then, as just was proved, $j \in K$. But in this case the inequality $\sum_\nu \vec{y}^\nu \geq \vec{0}$ cannot be valid, because no firms, except for the ones in - , produce commodities $j \in K$. If $w_j > 0$, then $j \in K$ and it is possible to produce commodity j without commodities $i \notin K$. This contradicts the indecomposability assumption. Thus $\vec{w} = \vec{0}$ and $\sum_{\nu \notin} \vec{y}^\nu \geq \vec{0}$. The firms $\nu \notin$ - can only use goods $j \in K$ as inputs $\sum_{\nu \notin} y^\nu_j \leq 0$, therefore $y^\nu_j = 0$ for all $j \in K$, $\nu \notin$ - . Hence firms $\nu \notin$ - produce output without using commodities $j \in K$. From (3) it follows that this is possible only if $K = \emptyset$ i.e. if $\vec{r} \in int\ R^n_+$. ■

PROOF OF THEOREM 1. Assume that the set of matrices $\{A^\nu\}$ is non-profitable. Taking any $\vec{p}_{BA} \neq \vec{0}$ and setting $I_{BA} = 0$, $\vec{c}_{BA} = \vec{x}^\nu_{BA} = \vec{v}^\nu_{BA} = \vec{z}^\nu_{BA} = \vec{0}$, we obtain a trivial BA-equilibrium. Indeed, solving the firms' optimization problems and applying the Walras' law we have $I_{BA} = 0$ for any $\vec{p}_{BA}$. If $I_{BA} = 0$ then consumers do not buy anything. So there are no non-trivial equilibria in this case.

Assume that the set of matrices $\{A^\nu\}$ is profitable. Let us introduce the supply correspondence $S^\nu(\vec{p}) = \{\vec{x}^\nu(\vec{p}) - \vec{v}^\nu(\vec{p})\}$, where $\vec{x}^\nu(\vec{p})$, $\vec{v}^\nu(\vec{p})$ are all possible solutions of the problem

$$\max \vec{p}A^\nu\vec{x}^\nu - \vec{p}\vec{v}^\nu \text{ s.to } \vec{x}^\nu - \vec{v}^\nu \leq \vec{y}^\nu \in Y^\nu \text{ and } \vec{v}^\nu \leq 2\vec{m},\ \vec{y}^\nu \geq -3\vec{m}. \tag{23}$$

The correspondence $S^\nu : P^n \to 2^{R^n}$ has non-empty compact images and a closed graph, because the objective function of the problem (23) is continuous, the admissible set is compact and does not depend on $\vec{p}$. The aggregate supply correspondence is $S(\vec{p}) = \sum_\nu S^\nu(\vec{p})$.

Profitability of the matrices $\{A^\nu\}$ implies that all firms achieve non-negative values of the objective function $\vec{p}A^\nu\vec{x}^\nu - \vec{p}\vec{v}^\nu$ with at least one firm achieving a strictly positive value. Therefore for any $\vec{w} \in S(\vec{p})$ $\vec{p}\vec{w} = \sum_\nu \vec{p}(\vec{x}^\nu - \vec{v}^\nu) \geq \sum_\nu(\vec{p}A^\nu\vec{x}^\nu - \vec{p}\vec{v}^\nu) > 0$. So $S(\vec{p})$ satisfies the conditions of the Lemma 1 below. Therefore there exist prices $\vec{p}_*$, a consumption vector $\vec{c}_*$, consumers' income I_* and solutions to the problems (23) $\vec{x}^\nu_*$, $\vec{v}^\nu_*$ and $\vec{y}^\nu_*$. The consumer rationality and Walras' law are satisfied.

Let us set $\vec{x}^\nu_{BA} = [\vec{x}^\nu_* - \vec{v}^\nu_*]_+$, $\vec{v}^\nu_{BA} = [\vec{x}^\nu_* - \vec{v}^\nu_*]_-$, $\vec{y}^\nu_{BA} = \vec{y}^\nu_*$. First, $\vec{x}^\nu_{BA}$, $\vec{v}^\nu_{BA}$,

$\vec{y}^{\nu}_{BA}$ along with $\vec{p}_*$, $\vec{c}_*$, I_* satisfy the Walras' law and the balance condition

$$\vec{c}_* \leq \sum_{\nu}(\vec{x}^{\nu}_{BA} - \vec{v}^{\nu}_{BA}).$$

Second, they are solutions to the problem (23). Third, they satisfy

$$\vec{x}^{\nu}_{BA} = [\vec{x}^{\nu}_* - \vec{v}^{\nu}_*]_+ \leq [\vec{y}^{\nu}_*]_+.$$

Hence,

$$\vec{m} \geq \sum_{\nu} \vec{x}^{\nu}_{BA} \geq \vec{c}_* + \sum_{\nu} \vec{v}^{\nu}_{BA}.$$

Thus, $2\vec{m} > \vec{v}^{\nu}_{BA}$ and $-3\vec{m} < \vec{y}^{\nu}_{BA}$. Thus the additional constraints $\vec{v}^{\nu} \leq 2\vec{m}$, $\vec{y}^{\nu} \geq -3\vec{m}$ in (23) can be dropped so that $\vec{x}^{\nu}_{BA}$, $\vec{v}^{\nu}_{BA}$, $\vec{y}^{\nu}_{BA}$ are solutions the firm's problem in the BA-equilibrium. By setting $\vec{p}_{BA} = \vec{p}_*$, $I_{BA} = I_*$, $\vec{c}_{BA} = \vec{c}_*$ we obtain the BA-equilibrium. ■

Lemma 1 *Assume that a set-valued correspondence $S : P^n \to 2^{R^n}$ (aggregate supply function) defined on the price simplex P^n has convex images, a closed graph and satisfies the condition $\vec{\xi}$*

$$\vec{p}\vec{\xi} > 0 \quad if \quad \vec{p} \in P^n, \vec{\xi} \in S(\vec{p}) \tag{24}$$

Then there exist equilibrium prices $\vec{p}_* \in P^n$, a supply vector $\vec{w}_*$ and a consumption vector $\vec{c}_*$ such that

$$\vec{\xi}_* \in S(\vec{p}_*), \quad \vec{c}_* \leq \vec{w}_*, \quad \vec{c}_* \in Arg \max_{\vec{c} \in R^n_+,\ \vec{p}_* \vec{c} \leq \vec{p} * \vec{\xi} *} U(\vec{c})$$

Proof. From (24) it follows that there exist numbers $I_+ \geq I_- > 0$ such that $\vec{p}\vec{\xi} \in J = [I_-, I_+]$ if $\vec{\xi} \in S(\vec{p})$. Suppose that consumers have some 'stock of money' $I \in J$ and define the demand function $D : P^n \times J \to 2^{R^n}$ as

$$D(\vec{p}, I) = \mathrm{Argmax}_{\vec{c}} U(\vec{c}) \quad \text{s.to} \quad \vec{0} \leq \vec{c} \leq 2\vec{m}, \quad \vec{p}\vec{c} \leq I, \tag{25}$$

where $\vec{m}$ – the upper bound of images of S.

This correspondence has non-empty, convex values. Since I is separated from zero it also has a closed graph. Images of the excess supply function $S(\vec{p}) - D(\vec{p}, I)$ are contained in some convex compact K.

Let us define another correspondence $A : P^n \times K \times I \to 2^{P^n \times K \times I}$ such that $\langle \vec{p}', \vec{e}', I' \rangle \in A(\vec{p}, \vec{e}, I)$ iff

$$\vec{p}' \in \mathrm{Argmax}_{\vec{q} \in P^n} \vec{q}\vec{e}, \ \vec{e}' = \vec{\xi} - \vec{c}, \ I' = \vec{p}\vec{\xi} \tag{26}$$

for some $\vec{\xi} \in S(\vec{p})$, $\vec{c} \in D(\vec{p}, I)$.

Correspondence A satisfies the conditions of the Kakutani theorem and therefore has a fixed point $\vec{p}_*, \vec{e}_*, I_* \in A(\vec{p}_*, \vec{e}_*, I_*)$. From (26) it follows that there exists $\vec{\xi}_* \in S(\vec{p}_*)$ such that $I_* = \vec{p}_*\vec{\xi}_*$. Let $\vec{c}_* \in D(\vec{p}_*, I_*)$. Due to (26) $\vec{p}_*\vec{c}_* \leq I_*$, so that $\vec{p}_*(\vec{\xi}_* - \vec{c}_*) = \vec{p}_*\vec{e}_* \geq 0$. For any $\vec{p} \in P^n$ it is true $\vec{p}\vec{e}_* \geq \vec{p}_*\vec{e}_*$. So $\vec{p}\vec{e}_* \geq 0$ for any $\vec{p} \in P^n$ and therefore $\vec{e}_* = \vec{\xi}_* - \vec{c}_* \geq 0$. Due to non-satiability of the demand, $\vec{c}_*$ lies on the Pareto frontier of plausible set in (25). Since $c_j < 2m_j$, we have $\vec{p}_*\vec{c}_* = I_* = \vec{p}_*\vec{\xi}_*$. ■

Proof of Proposition 2. In a BA-equilibrium, firms effectively maximize profit in prices $\vec{p}^\nu$ where $p_i^\nu = p_{BA\,i}a_i^\nu$ for $i = i_\nu$ and $p_i^\nu = p_{BA\,i}$ for $i \neq i_\nu$. Hence, $\vec{y}_{BA}^\nu \in \partial Y^\nu$.

Suppose that BA-equilibrium is efficient in terms of Definition 3. It means that for some $\vec{r} \in R_{++}^n$ and $\vec{y} = \sum_\nu \vec{y}_{BA}^\nu$ holds $\vec{r}\vec{y} = \max_{\vec{y}^\nu \in Y^\nu} \sum_\nu \vec{r}\vec{y}^\nu = \sum_\nu \vec{r}\vec{y}_{BA}^\nu$. Therefore $\vec{r}\vec{y}_{BA}^\nu = \max_{\vec{y}^\nu \in Y^\nu} \vec{r}\vec{y}^\nu$ or all firms maximize profit measured in the same prices $\vec{r}$. Due to differentiability of the production functions, there is a unique vector normal to the frontier of Y^ν at the point $\vec{y}_{BA}^\nu$, so $\vec{r} = \vec{p}^\nu$ for all ν. Take i and ν such that $p_{BA\,i}y_{BA}^\nu < 0$. From $\sum_\nu y_{BA\,i}^\nu \geq 0$ it follows that there exists a firm μ, which produce this commodity $y_{BAi}^\mu > 0$. Therefore $r_i = p_{BA\,i}a_i^\mu < p_{BA\,i} = r_i$. Proved by contradiction. ■

Proof of Proposition 3. The firm's optimization problem has a finite solution which is a saddle point of the Lagrange function

$$L^\nu = \vec{p}_Q A^\nu \vec{x}^\nu - \vec{p}_Q \vec{v}^\nu + \vec{h}^\nu(\vec{y}^\nu - \vec{x}^\nu + \vec{v}^\nu - \vec{t}^\nu) + \theta^\nu \vec{q}\vec{t}^\nu,$$

where $\vec{h}^\nu \geq \vec{0}$, $\theta^\nu \geq 0$ are Lagrange multipliers for the constraints (14), (15) respectively. Since there are no more constraints on $\vec{t}^\nu$, $\vec{h}^\nu = \theta^\nu \vec{q}$. Thus

$$\inf_{\vec{r}^\nu \geq \vec{0}, \theta^\nu \geq 0} \; \sup_{\vec{x}^\nu, \vec{v}^\nu \geq \vec{0}; \vec{t}^\nu; \vec{y}^\nu \in Y^\nu} L^\nu = \min_{\theta^\nu : \vec{p}A^\nu \leq \theta^\nu \vec{q} \leq \vec{p}} [\theta^\nu \max_{\vec{y}^\nu \in Y^\nu} \vec{q}\vec{y}^\nu].$$

The firm maximizes output in prices $\vec{q}$. Second, since $\max_{\vec{y}^\nu \in Y^\nu} \vec{q}\vec{y}^\nu \geq 0$, we can find θ^ν in the saddle point

$$\theta^\nu = \max_{i: p_i \neq 0} \frac{p^i a_i^\nu}{q^i} > 0,$$

and the following inequality should hold

$$\theta^\nu \vec{q} \leq \vec{p}.$$

■

Proof of Theorem 2. If Q-equilibrium exists, it is efficient since all firms maximize output in the same prices $\vec{q}_Q$.

In order to prove existence, we shall use the Gale lemma (Nikaido (1968)) which is in turn based on the Kakutani theorem. Let us build a set-valued excess demand correspondence of the price simplex $P^n = \{\vec{q} : q_i \geq 0, \sum_i q_i = 1\}$ onto

$2^{\mathbf{R}^n}$. Take an arbitrary $\vec{q} \in P^n$. For this $\vec{q}$, introduce a set-valued correspondence $\vec{y}(\vec{q})$ =Argmax$_{\vec{y}\in Y}\, \vec{q}\vec{y}$. Then introduce the cash price in the following way. If $\vec{y} \in \mathbf{R}^n_+$ take $\vec{p}(\vec{q}) = \{\vec{p} \in P^n : \forall \vec{y}' \geq \vec{0} : \vec{p}\vec{y}' \leq \vec{p}\vec{y} \Rightarrow U(\vec{y}) \geq U(\vec{y}')\}$. If $\vec{y} \notin \mathbf{R}^n_+$ then $\vec{p}(\vec{q}) = \{\vec{p} \in P^n : p_i = 0\, \forall i : y_i > 0\}$. This correspondence is upper-hemicontinuous and has a closed graph. (In other words, $\vec{p}(\vec{q})$ is the price vector under which consumers choose $\vec{y}(\vec{q})$ if $\vec{y}(\vec{q}) \in \mathbf{R}^n_+$. For $\vec{y}(\vec{q}) \notin \mathbf{R}^n_+$ we complete the definition of $\vec{p}(\vec{q})$ so that the correspondence remains upper-hemicontinuous.)

Let us also consider the set of solutions $\vec{x}^\nu(\vec{q})$,$\vec{v}^\nu(\vec{q})$, $\vec{t}^\nu(\vec{q})$, $\vec{y}^\nu(\vec{q})$ of the problem of maximizing (13) subject to the constraints (14)-(15) and two auxiliary constraints

$$-\vec{M} \leq \vec{t}^\nu, \tag{27}$$
$$\vec{v}^\nu \leq \vec{m} \tag{28}$$

under given prices $\vec{q}$ and $\vec{p} \in \vec{p}(\vec{q})$. Here $\vec{M} = 2N\vec{m} + \sum_\nu \vec{m}^\nu$, and $m^\nu_{i_\nu} = m^\nu$ and $m^\nu_i = 0$ for $i \neq i_\nu$.

The constraints (27)-(28) make the optimization problem compact for all $\vec{p}, \vec{q} \in P^n$. Indeed, using (14), (27) and (28), we obtain $\vec{0} \leq \vec{v}^\nu \leq \vec{m}$ and $\vec{0} \leq \vec{x}^\nu \leq 2\vec{m} + \vec{M}$. Similarly, $-\vec{M} \leq \vec{t}^\nu \leq 2\vec{m}$. The value of the objective function is bounded from above: $\vec{p}A^\nu\vec{x}^\nu - \vec{p}\vec{v}^\nu \leq \vec{p}(\vec{x}^\nu - \vec{v}^\nu) \leq \vec{p}(\vec{y}^\nu - \vec{t}^\nu) \leq \Pi^\nu(\vec{p}) + \vec{M}$.

Let us introduce the excess demand correspondence $\Xi(\vec{q}) : P^n \to 2^{\mathbf{R}^n}$. For a given $\vec{q}$, $\Xi(\vec{q})$ includes vectors $\vec{\xi}$ such that $\vec{\xi} = -\sum_\nu \vec{t}^\nu$ where $\vec{t}^\nu$ are solutions of the optimization problem above under given $\vec{q}$ and $\vec{p}$, where $\vec{p} \in \vec{p}(\vec{q})$. Apparently, $\Xi(\vec{q})$ satisfies the Walras' law: adding up constraints (15) for all ν we obtain $\vec{q}\vec{\xi} = -\sum_\nu \vec{q}\vec{t}^\nu \leq 0$. The correspondence satisfies all the conditions of the excess correspondence in the Gale Lemma (Nikaido (1968)). Therefore there exists such $\vec{q}^* \in P^n$ and $\vec{\xi}^*$ that $\vec{\xi}^* \in \Xi(\vec{q}^*)$ and $\vec{\xi}^* \leq \vec{0}$. Let us denote $\vec{p}^*, \vec{x}^{\nu*}$,$\vec{v}^{\nu*}, \vec{t}^{\nu*}, \vec{y}^{\nu*}, \vec{y}^*$ the respective values of $\vec{p}(\vec{q}), \vec{x}^\nu(\vec{q})$,$\vec{v}^\nu(\vec{q}), \vec{t}^\nu(\vec{q}), \vec{y}^\nu(\vec{q}), \vec{y}(\vec{q})$.

Now, we shall construct Q-equilibrium using $\vec{q}^*$. First, let us prove that the constraint (27) is not binding. Indeed, suppose that $t^{\nu*}_i = -M_i$ for some i and ν. Therefore $\sum_{\mu\neq\nu} t^{\mu*}_i \geq M_i$. On the other hand, adding up constraints (14) for all $\mu \neq \nu$, we obtain $\sum_{\mu\neq\nu} t^{\mu*}_i \leq \sum_{\mu\neq\nu} y^{\mu*}_i + \sum_{\mu\neq\nu} v^{\mu*}_i - \sum_{\mu\neq\nu} x^{\mu*}_i \leq \sum_\nu m^\nu + 2(N-1)m_i < M_i$. The contradiction proves that $t^{\nu*}_i > -M_i$ for all i and ν. The Lagrange function is therefore as follows

$$L^\nu = (\vec{p}A^\nu\vec{x}^\nu - \vec{p}\vec{v}^\nu) + \vec{h}^\nu(\vec{y}^\nu - \vec{x}^\nu + \vec{v}^\nu - \vec{t}^\nu) + \theta^\nu\vec{q}\vec{t}^\nu - \vec{l}\vec{v}^\nu + \vec{l}\vec{m}$$

where $\vec{h}^\nu$, θ^ν, $\vec{l} \geq 0$ are the Lagrange multipliers for the constraints (14),(15), and (28), respectively. Since $\vec{t}^\nu$ is effectively unconstrained, $\vec{h}^\nu = \theta^\nu\vec{q}^*$, and $\theta^\nu > 0$, so that constraint (15) is binding. All firms choose output that maximizes profit in prices $\vec{q}^*$: $\vec{y}^{\nu*} \in$Argmax$_{\vec{y}\in Y^\nu}\, \vec{q}^*\vec{y}$. Therefore $\vec{y}^* = \sum_\nu \vec{y}^{\nu*}$.

Let us show that $\vec{q}^* > 0$. Indeed, admit that $q^*_i = 0$ for some i. Proposition 1 implies $\vec{y}^* \notin \mathbf{R}^n_+$. There can be two cases. First, consider the case $p^*_i > 0$. In this case each firm would make an infinite profit by increasing x_i^ν and decreasing t_i^ν by the same amount. Thus there can only be the case $p_i^* = q_i^* = 0$. By construction of $\vec{p}(\vec{q})$, $p_i^* = 0$ implies $y_i^* > 0$. But this in turns requires $q_i^* > 0$. Indeed, each firm maximizes output in prices $\vec{q}^*$, so that if $q_i^* = 0$ were the case, firms with $i_\nu = i$ would not want to produce a positive amount of the good i.

Now let us prove that $\vec{y}^* \in \mathbf{R}_+^n$. Since $q_i^* > 0$ for all i, $\sum_\nu \vec{t}^{\nu *} = 0$. Therefore $\sum_\nu (\vec{x}^{\nu *} - \vec{v}^{\nu *}) = \vec{y}^*$. Suppose that $\vec{y}^* \notin \mathbf{R}_+^n$. Then by construction of $\vec{p}^*$, $\vec{p}^* \vec{y}^* < 0$. On the other hand $\vec{p}^* \vec{y}^* = \sum_\nu (\vec{p}^* \vec{x}^{\nu *} - \vec{p}^* \vec{v}^{\nu *}) \geq \sum_\nu (\vec{p}^* A^\nu \vec{x}^{\nu *} - \vec{p}^* \vec{v}^{\nu *}) \geq 0$.

Thus $\vec{y}^* \in \mathbf{R}_+^n$. By definition of $\vec{p}^*$,

$$\vec{y}^* \in \operatorname{Arg} \max_{\vec{p}^* \vec{c} \leq \sum_\nu (\vec{p}^* \vec{x}^{\nu *} - \vec{p}^* \vec{v}^{\nu *}),\, \vec{c} \geq 0} U(\vec{c})$$

To construct Q-equilibrium, take $\vec{q}_Q = \vec{q}^*$, $\vec{p}_Q = \vec{p}^*$, $\vec{c}_Q = \vec{y}_Q = \vec{y}^*$, $\vec{t}_Q^\nu = \vec{t}^{\nu *}$, $\vec{y}_Q^\nu = \vec{y}^{\nu *}$, $\vec{x}_Q^\nu = [\vec{x}^{\nu *} - \vec{v}^{\nu *}]_+$, $\vec{v}_Q^\nu = [\vec{x}^{\nu *} - \vec{v}^{\nu *}]_-$. To complete the proof we need to show that constraint (28) is not binding.

Since $\vec{x}^{\nu *}, \vec{v}^{\nu *}$ are solutions of the firm's optimization problem, then $\vec{x}_Q^\nu$, $\vec{v}_Q^\nu$ are also solutions. Indeed, if we replace $\vec{x}^{\nu *}, \vec{v}^{\nu *}$ with $\vec{x}_Q^\nu$, $\vec{v}_Q^\nu$ then constraints will still be met ($\vec{x}_Q^\nu - \vec{v}_Q^\nu = \vec{x}^{\nu *} - \vec{v}^{\nu *}$) while the objective function can only increase ($A^\nu \leq E$). Suppose that (28) is binding for some i and ν then $x_{Q\,i}^\nu = 0$ and $v_{Q\,i}^\nu = m_i$. Hence the Lagrange multipliers must satisfy and $l_i = \theta^\nu q_i - p_i$ and $p_i a_i^\nu - \theta^\nu q_i \leq 0$. Hence $l_i \leq p_i(a_i^\nu - 1) \leq 0$. By definition, $l_i \geq 0$. Thus, $l_i = 0$ so that by lifting the constraint (28) we do not change the value of the objective function.

We have proved the existence of Q-equilibrium. In this equilibrium $\vec{q}_Q > \vec{0}$ and $\vec{y}_Q \geq \vec{0}$. The last statement of the Theorem follows from Proposition 3. ■

Proof of Proposition 4. Let us consider the following optimization problem

$$\max_{\vec{x}^\nu, \vec{v}^\nu, \vec{t}^\nu, \vec{y}^\nu} \sum_\nu (\vec{p} A \vec{x}^\nu - \vec{p} \vec{v}^\nu) \tag{29}$$

subject to the individual firms' technology constraints (14), the balance constraint for the quasi-money market (16), and an auxiliary constraint

$$\sum_\nu \vec{v}^\nu \leq \vec{m}. \tag{30}$$

The latter makes the problem compact. Indeed, (30) implies $\vec{v}^\nu \leq \vec{m}$ and therefore $\sum_\nu (\vec{x}^\nu + \vec{t}^\nu) \leq 2\vec{m}$. Then $\sum_\nu \vec{x}^\nu \leq 2\vec{m}$ and $\vec{x}^\nu \leq 2\vec{m}$. Using (14), one obtains $\vec{t}^\nu \leq 2\vec{m}$ and $-\vec{t}^\nu \leq \sum_{\mu \neq \nu} \vec{t}^\mu \leq 2N\vec{m}$. The Slater condition is also satisfied (take $v_{i_\nu}^\nu = 0$, $v_i^\nu = -2m_i^\nu/3N$, $i \neq i_\nu$; $y_{i_\nu}^\nu = f^\nu(\vec{m}/3N)$, $y_i^\nu = -m_i^\nu/3N$, $i \neq i_\nu$; $x_{i_\nu}^\nu = t_{i_\nu}^\nu = y_{i_\nu}^\nu/3$, $x_i^\nu = t_i^\nu = 0$, $i \neq i_\nu$). The Lagrange function is as follows

$$L = \sum_\nu \left[(\vec{p} A \vec{x}^\nu - \vec{p} \vec{v}^\nu) + \vec{h}^\nu (\vec{y}^\nu - \vec{x}^\nu + \vec{v}^\nu - \vec{t}^\nu) + \vec{g} \vec{t}^\nu - \vec{l} \vec{v}^\nu \right] + \vec{l} \vec{m} \tag{31}$$

where $\vec{h}^\nu$, $\vec{g}$, $\vec{l} \geq 0$ are the Lagrange multipliers for the constraints (14),(16), (30) respectively. Denote $\vec{h}_*^\nu$, $\vec{g}_*$, $\vec{l}_*$, $\vec{x}_*^\nu$, $\vec{v}_*^\nu$, $\vec{y}_*^\nu$, $\vec{t}_*^\nu$ the saddle point of (31). Since the solution is finite, and $\vec{t}^\nu$ is unconstrained, $\vec{h}_*^\nu = \vec{g}_*$ for all ν, and

$$\vec{p}A\vec{x}_*^\nu = \vec{g}_Q\vec{x}_*^\nu, \;\; \vec{p}\vec{v}_*^\nu = \vec{g}_*\vec{v}_*^\nu, \;\; \vec{g}_*(\vec{x}_*^\nu - \vec{v}_*^\nu) = \vec{g}_*\vec{y}_*^\nu, \; \vec{g}_*\sum_\nu \vec{t}_*^\nu = 0, \tag{32}$$

$$\inf_{\vec{h}^\nu, \vec{g}, \vec{f} \geq \vec{0}} \; \sup_{\vec{x}^\nu, \vec{v}^\nu \vec{f} \geq \vec{0}, \vec{y}^\nu \in Y^\nu, \vec{t}^\nu} L = \min_{\alpha_\nu \vec{p}A^\nu \leq \vec{g}} \left[\vec{l}_*\vec{m} + \max_{\vec{y}^\nu \in Y^\nu} \vec{g}\sum_\nu \vec{y}^\nu \right] \tag{33}$$

where

$$l_{*i} = [g_{*i} - p_i]_+ \tag{34}$$

Let us introduce the aggregate supply correspondence $S(\vec{p}) = \{\sum_\nu(\vec{x}_*^\nu - \vec{v}_*^\nu)\}$ where $\vec{x}_*^\nu, \vec{v}_*^\nu$ are solutions of the optimization problem. The conditions (32) and (33) imply $\vec{p}\sum_\nu(\vec{x}_*^\nu - \vec{v}_*^\nu) \geq \sum_\nu(\vec{p}A\vec{x}_*^\nu - \vec{p}\vec{v}_*^\nu) = \vec{g}_*\sum_\nu(\vec{x}_*^\nu - \vec{v}_*^\nu) = \vec{g}_*\sum_\nu \vec{y}_*^\nu = \max_{\vec{y}^\nu \in Y^\nu} \vec{g}_*\sum_\nu \vec{y}^\nu$.

Since $\vec{g}_* \in R_{++}^n$, we can apply Proposition 1. Hence $\max_{\vec{y}^\nu \in Y^\nu} \vec{g}_*\sum_\nu \vec{y}^\nu > 0$ and $\vec{p}S(\vec{p}) > 0$ for all $\vec{p}$. Thus the aggregate supply function S satisfies the conditions of the Lemma 1 in the Appendix and we obtain existence of equilibrium prices $\vec{p}_Q$, firms' production, sales and purchases vectors $\vec{y}_*^\nu, \vec{x}_*^\nu, \vec{v}_*^\nu$, consumption vector $\vec{c}_Q \geq \vec{0}$, and income $I_Q = \vec{p}_Q\vec{c}_Q$ that satisfy consumer rationality condition (6), balance condition (5), and Walras' law (7).

From now on we will fix $\vec{p} = \vec{p}_Q$ and will use the saddle point of the Lagrange function (31) under $\vec{p} = \vec{p}_Q$ to complete construction of Q-equilibrium.

The balance condition (5) implies $\sum_\nu \vec{y}_*^\nu \geq \vec{0}$. Therefore, in order to minimize (33) with regard to $\vec{g}$, we should set $\vec{g}$ as low as possible, i.e. $\vec{g}_* = \vec{p}_QA$. Hence,

$$g_*^i = a_i p_Q^i. \tag{35}$$

Since Proposition 1 requires $\vec{g}_*$ to be positive, the cash prices $\vec{p}_Q$ must also be positive. Let us now find such quasi-monetary prices $\vec{q}_Q$ that $\vec{y}_Q^\nu = \vec{y}_*^\nu$ are solutions of individual firms' problems (13)-(15). Proposition 3 implies that this is the case only if $\vec{q}_Q = \lambda\vec{g}_Q$. Let us set $\vec{q}_Q = \lambda\vec{p}_QA$. In this case $\theta^\nu = \max_{i:q_Q^i \neq 0} \frac{p_Q^i a_i^\nu}{q_Q^i} = 1$ and $\vec{l}_Q = 0$ (the latter allows to drop the auxiliary constraint (30)). Under $\vec{q}_Q = \lambda\vec{p}_QA$, firms choose output $\vec{y}_Q^\nu = \vec{y}_*^\nu$ and are indifferent between which to sell in the monetary market and whether to sell in the monetary or quasi-monetary market. Let us find $\vec{x}_Q^\nu, \vec{v}_Q^\nu, \vec{t}_Q^\nu$ such that they solve both the maximization problem (29) and the individual firms' problems. Take an arbitrary $\vec{x}_*^\nu, \vec{v}_*^\nu, \vec{t}_*^\nu$ and calculate $\Delta^\nu = \vec{q}_Q\vec{t}_*^\nu$. If $\Delta^\nu < 0$ then constraint (15) is not satisfied and we need to increase $\vec{t}^\nu$ and decrease $\vec{x}^\nu$ by the same amount. Let us decrease x_1^ν by Δ^ν/q_1 and increase t_1^ν by Δ^ν/q_1. Doing this for all ν we obtain $x_{o1}^\nu = x_{*1}^\nu - \Delta^\nu/q_1$, $t_{o1}^\nu = t_{*1}^\nu + \Delta^\nu/q_1$ and $x_{oi}^\nu = x_{*i}^\nu$, $t_{oi}^\nu = t_{*i}^\nu$ for $i \neq 1$. Since $\sum_\nu \Delta^\nu = 0$, the aggregate supply in money and quasi-money market for the good 1 will remain the same $\sum_\nu(\vec{x}^\nu_o - \vec{v}^\nu_*) = \sum_\nu(\vec{x}^\nu_* - \vec{v}^\nu_*)$ and $\sum_\nu \vec{t}^\nu_o = \sum_\nu \vec{t}^\nu_*$. Now let us set $\vec{x}^\nu_Q = [\vec{x}^\nu_o - \vec{v}^\nu_*]_+$, $\vec{v}^\nu_Q = [\vec{x}^\nu_o - \vec{v}^\nu_*]_-$, $\vec{t}^\nu_Q = \vec{t}^\nu_o$.

Hence $\left\langle \vec{p}_Q, \vec{q}_Q, \vec{c}_Q, \{\vec{x}^\nu_Q\}, \{\vec{v}^\nu_Q\}, \{\vec{t}^\nu_Q\}, I_Q \right\rangle$ is a Q-equilibrium. ■

Proof of Proposition 6. We shall prove that in the case $\vec{s} = \vec{p}$ the maximum value of the firm's objective function is $\hat{J} = a^\nu_{i_\nu} \max_{\vec{y}^\nu \in Y^\nu} \vec{p}\vec{y}^\nu$. Indeed, if $\vec{s} = \vec{p}$

$$J \le a^\nu_{i_\nu}\vec{p}(\vec{x}^\nu - \vec{v}^\nu) \le a^\nu_{i_\nu}\vec{p}(\vec{x}^\nu - \vec{v}^\nu + \vec{z}^\nu - \vec{w}^\nu) \le a^\nu_{i_\nu}\vec{p}\vec{y}^\nu \le \hat{J}.$$

On the other hand the firm can reach $\hat{J}$ e.g. via the following strategy:

- $\vec{v}^\nu = 0$;
- for $i \ne i_\nu$ set $w^\nu_i = -y^\nu_i$ and $x^\nu_i = z^\nu_i = 0$;
- for $i = i_\nu$ set $z^\nu_i = -\left[\sum_{j \ne i_\nu} p_j y^\nu_j\right]_+ / p_i$ and $w^\nu_i = -\left[\sum_{j \ne i_\nu} p_j y^\nu_j\right]_+ / p_i$ and $x^\nu_i = y^\nu_i - z^\nu_i + w^\nu_i$ (the latter expression is non-negative since $\max\limits_{\vec{y}^\nu \in Y^\nu} \langle \vec{p}\vec{y}^\nu \rangle \ge 0$.

The output is always Walrasian whenever $\vec{s} = \vec{p}$.

In order for the supply function to be equal to the one in the BA-economy we need to require

$$\frac{(p_{i_\nu} - s_{i_\nu})a^\nu_{i_\nu} + \lambda^\nu s_{i_\nu}}{(p_j - s_j) + \lambda^\nu s_j} = \frac{p_{i_\nu} a^\nu_{i_\nu}}{p_j}$$

for all $i \ne i_\nu$. The condition $\vec{s} = 0$ is sufficient. If the equation holds for *all* firms, i.e. for all pairs of i and j then $\vec{s} = 0$ is necessary. ■

Proof of Theorem 3. Denote G^n the set of non-negative diagonal matrixes with unit trace. Fix an arbitrary $\chi \in (0, 1]$, set $\vec{s} = \chi\vec{p}G$, $G \in G^n$ and consider the problem of maximization of (17) subject to (18)-(20) and auxiliary constraints

$$\vec{w}^\nu + \vec{v}^\nu \le 2N\vec{m}, \quad \vec{y}^\nu \ge -3N\vec{m}. \tag{36}$$

The latter make the admissible set compact.

Consider a correspondence $S : P^n \times G^n \to 2^{R^n \times R^n \times R^1}$ such that $\langle \vec{u}, \vec{e}, \pi \rangle \in S(\vec{p}, G)$ if

$$\begin{aligned} \vec{u} &= \sum_\nu (\vec{x}^\nu - \vec{v}^\nu + \vec{z}^\nu - \vec{w}^\nu), \quad \vec{e} = \sum_\nu (\vec{z}^\nu - \vec{w}^\nu), \\ \pi &= \sum_\nu (\vec{p}(\vec{x}^\nu - \vec{v}^\nu) + (\vec{p} - \chi\vec{p}G)(\vec{z}^\nu - \vec{w}^\nu)), \end{aligned} \tag{37}$$

where $\vec{x}^\nu, \vec{v}^\nu, \vec{z}^\nu, \vec{w}^\nu, \vec{y}^\nu$ are maximizers of (17) subject to (18)-(20), (36). Lemma 3 implies that the correspondence S has non-empty compact convex images and closed graph. Lemma 3 also shows also that the optimization problem above satisfies Slater condition for the constraints (18), (19). According to the Kuhn-Tucker theorem, there exist such $\vec{r}^\nu$, θ^ν_C (Lagrange multipliers for these constraints) that

$$\begin{gathered}\vec{y}^\nu \in \operatorname{Arg\,max}_{\vec{y}\in Y^\nu, \vec{y}\geq -3N\vec{m}}\ \vec{r}^\nu\vec{y}, \quad \vec{r}^\nu \in \operatorname{Arg\,min}_{\vec{r}\geq\vec{p}A^\nu}\ \vec{r}\left(\vec{y}^\nu+\vec{v}^\nu+\vec{w}^\nu\right), \\ \theta^\nu_C \geq 0, \quad \vec{p}G\vec{z}^\nu - \vec{p}G\vec{w}^\nu \geq 0, \ \ \theta^\nu_C\left(\vec{p}G\vec{z}^\nu - \vec{p}G\vec{w}^\nu\right) = 0, \\ (\vec{p}-\chi\vec{p}G)A^\nu - \vec{r}^\nu + \theta^\nu_C\chi\vec{p}G \leq \vec{0}, \\ \vec{z}^\nu \geq \vec{0},\ \left((\vec{p}-\chi\vec{p}G)A^\nu - \vec{r}^\nu + \theta^\nu_C\chi\vec{p}G\right)\vec{z}^\nu = \vec{0}\end{gathered} \tag{38}$$

If $\theta^\nu_A > 0$ then (38) implies $\vec{p}G\vec{z}^\nu - \vec{p}G\vec{w}^\nu = 0$. If $\theta^\nu_A = 0$ then (38) and (38) imply $\vec{p}G\vec{z}^\nu = 0$. Using (38) we also obtain $\vec{p}G\vec{w}^\nu = 0$. So in any case $\vec{p}G\vec{z}^\nu - \vec{p}G\vec{w}^\nu = 0$ and by definition (37)

$$\vec{p}G\vec{e} = 0 \quad \text{whenever} \quad \langle\vec{u},\vec{e},\pi\rangle \in S(\vec{p},G)$$

Apparently, $\vec{x}^\nu = [\vec{y}^\nu]_+$, $\vec{v}^\nu = [\vec{y}^\nu]_-$, $\vec{z}^\nu = \vec{w}^\nu = \vec{0}$ satisfies constraints (18)-(20), (36). Therefore the optimum value of the objective function (17) is greater than or equal to $\max\limits_{\vec{0}\leq\vec{w}\leq\vec{m}} a^\nu_{i_\nu} p_{i_\nu} f^\nu(\vec{w}) - \vec{p}\vec{w}$. Since the economy is productive (2) and the matrices are profitable (12)

$$\pi \geq J_- > 0 \quad \text{if} \quad \langle\vec{u},\vec{e},\pi\rangle \in S(\vec{p},G), \quad \vec{p}\in P^n, G\in G^n.$$

Because the graph of S is closed, there exist a convex compact $K \subset R^n$ and $I \subset R^1_{++}$ such that $S(\vec{p},G) \subseteq K\times K\times I$.

As in the Proof of Lemma 1 we shall introduce 'money balance' ψ and define on $P^n\times I$ a demand function $D(\vec{p},\psi)$ with compact, convex, non-empty images and closed graph:

$$D(\vec{p},\psi) = \operatorname{Arg}\max_{\vec{c}} U(\vec{c}) \quad \text{s.to} \quad \vec{0}\leq\vec{c}\leq 2\vec{m}, \quad \vec{p}\vec{c}\leq\psi.$$

Finally, denote $Q = K - \{\vec{c} : \vec{0}\leq\vec{c}\leq\vec{m}\}$ and introduce a correspondence A_χ of the compact $Q\times K\times I\times P^n\times G^n$ onto itself: $\langle\vec{e}'_0,\vec{e}'_1,\psi',\vec{p}',G'\rangle \in A_\chi(\vec{e}_0,\vec{e}_1,\psi,\vec{p},G)$ if

$$\begin{aligned}\vec{e}'_0 &= \vec{u}-\vec{c}, \quad \vec{e}'_1 = \vec{e}, \quad \psi' = \pi \quad \text{for some } \langle\vec{u},\vec{e},\pi\rangle\in S(\vec{p},G), \quad \vec{c}\in D(\vec{p},\psi), \\ \vec{p}' &\in Argmax_{\vec{p}\in P^n}\vec{p}\vec{e}_0, \quad G' \in Argmax_{G\in G^n}\vec{p}G\vec{e}_1\end{aligned}$$

The correspondence A_χ satisfies all the conditions of Kakutani theorem, so it has a fixed point $\langle\vec{e}^\chi_0,\vec{e}^\chi_1,\psi_\chi,\vec{p}_\chi,G_\chi\rangle$:

$$\begin{gathered}\langle\vec{u}_\chi,\vec{e}^\chi_0,\psi_\chi\rangle \in S(\vec{p}_\chi,G_\chi), \quad \vec{e}^\chi_0 = \vec{u}_\chi - \vec{c}_\chi = \textstyle\sum_\nu(\vec{x}^\nu_\chi - \vec{v}^\nu_\chi + \vec{z}^\nu_\chi - \vec{w}^\nu_\chi) - \vec{c}_\chi, \\ \vec{c}_\chi \in D(\vec{p}_\chi,\psi_\chi), \quad \vec{e}^\chi_1 = \textstyle\sum_n(\vec{y}^\nu_\chi - \vec{w}^\nu_\chi), \\ \vec{p}_\chi G\vec{e}^\chi_1 \geq \vec{p}_\chi G_\chi\vec{e}^\chi_1 = 0 \text{ for all } G\in G^n, \\ \psi_\chi = \textstyle\sum_\nu(\vec{p}_\chi(\vec{x}^\nu_\chi - \vec{v}^\nu_\chi) + (\vec{p}_\chi - \chi\vec{p}_\chi G)(\vec{z}^\nu_\chi - \vec{w}^\nu_\chi)) = \vec{p}_\chi\vec{u}_\chi - \chi\vec{p}_\chi G_\chi\vec{e}^\chi_1 \geq \psi_- \\ \vec{p}\vec{e}^\chi_0 \geq \vec{p}_\chi\vec{e}^\chi_0 \ \text{ for all } \vec{p}\in P^n.\end{gathered} \tag{39}$$

The first line implies that $\vec{x}^\nu_\chi$, $\vec{v}^\nu_\chi$, $\vec{z}^\nu_\chi, \vec{w}^\nu_\chi$, $\vec{y}^\nu_\chi$ maximize (17) subject to (18)-(20), (36) with $\vec{p} = \vec{p}_\chi$, $\vec{s} = \chi\vec{p}_\chi G_\chi$.

We can now construct a C-equilibrium. Let us set $\vec{p}_C = \vec{p}_\chi$, $\vec{s}_C = \chi\vec{p}_\chi G_\chi$, $\psi_C = \psi_\chi$, $\vec{c}_C = \vec{c}_\chi$, $\vec{y}^\nu_C = \vec{y}^\nu_\chi$. By definition of the demand function (25) we have $\vec{p}_C\vec{c}_C \le \psi_C$, and (39) implies $\vec{p}_C\vec{e}^\chi_0 \ge 0$. Therefore, $\vec{e}^\chi_0 \ge \vec{0}$, or due to (19),

$$\vec{c}_C \le \sum_\nu(\vec{x}^\nu_\chi - \vec{v}^\nu_\chi + \vec{z}^\nu_\chi - \vec{w}^\nu_\chi) \le \sum_\nu \vec{y}^\nu_\chi$$

Because the right-hand side is not greater than $\vec{m}$, no component of $\vec{c}_C$ is equal to $2m_i$, so

$$\vec{p}_C\vec{c}_C = \psi_C > 0, \quad \vec{c}_C \in \operatorname{Arg}\max_{\vec{c}\in R^n_+}\{U(\vec{c})\} \quad \text{s.to} \quad \vec{p}_C\vec{c} \le \psi_C.$$

Therefore $\vec{c}_C \in R^n_{++}$, so that the constraint $\vec{y} \ge -3N\vec{m}$ in the first line of (38) can be ignored. Then Lemma 4 below implies $\vec{p}_C > \vec{0}$. Using this fact and (39), one can get $\vec{e}^\chi_1 \ge \vec{0}$. In other words, (21) holds for $\vec{z}^\nu_\chi, \vec{w}^\nu_\chi$. Notice that the inequality (21) can be strict only for such i that $g^i_\chi = 0$.

If $g^i_\chi = 0$, then $s^i_C = 0$ and cash and arrears markets become equivalent. If $s^i_C = 0$ then all exchanges of the good i can be implemented in the cash market keeping all the constraint met. Moreover, because $\vec{p}_C \ge \vec{s}_C$ it is possible to subtract common positive components from the pairs of the vectors $\vec{x}^\nu_\chi$, $\vec{v}^\nu_\chi$ and $\vec{z}^\nu_\chi, \vec{w}^\nu_\chi$, without a decrease in the objective functions and keeping all constraints intact. Namely, we can set $x^\nu_{C\,i} = [x^\nu_{\chi i} - v^\nu_{\chi i}]_+$, $v^\nu_{C\,i} = [x^\nu_{\chi i} - v^\nu_{\chi i}]_-$, $z^\nu_{C\,i} = [z^\nu_{\chi i} - w^\nu_{\chi i}]_+$, $w^\nu_{C\,i} = [z^\nu_{\chi i} - w^\nu_{\chi i}]_-$ if $s^i_C > 0$ and $x^\nu_{C\,i} = [x^\nu_{\chi i} - v^\nu_{\chi i} + z^\nu_{\chi i} - w^\nu_{\chi i}]_+$, $v^\nu_{C\,i} = [x^\nu_{\chi i} - v^\nu_{\chi i} + z^\nu_{\chi i} - w^\nu_{\chi i}]_-$, $z^\nu_{Ci} = w^\nu_{Ci} = 0$ if $s^i_C = 0$.

Since $\sum_\nu \vec{z}^\nu_A = \sum_\nu \vec{w}^\nu_A$, the exchange vectors defined above satisfy all the conditions of C-equilibrium. In each pair of non-negative vectors $\vec{x}^\nu_C$, $\vec{v}^\nu_C$ and $\vec{y}^\nu_C$, $\vec{u}^\nu_C$ only one vector can be positive, so that constraint (19) implies $\vec{x}^\nu_C \le \vec{m}$ and $\vec{y}^\nu_C \le \vec{m}$. Therefore the auxiliary constraints (36) can be dropped and the exchange vectors $\vec{x}^\nu_C, \vec{v}^\nu_C, \vec{z}^\nu_C, \vec{w}^\nu_C, \vec{y}^\nu_C$ solve the original problem (17), (18)-(20). ■

Lemma 2 *Let K be a convex compact in R^m such that $\vec{0}$ $\in$int K. Then a correspondence $R : R^n \to 2^K$, $R(\vec{\xi}) = \{\vec{x} \in K : \vec{\xi}\vec{x} \ge 0\}$ is continuous in terms of Hausdorf (and, therefore, upper- and lower-hemicontinuous) if $\vec{\xi} \ne \vec{0}$.*

Proof. Denote $\phi(\vec{q})$ and $\psi_{\vec{\xi}}(\vec{q})$ the support functions of the sets K and $R(\vec{\xi})$ respectively. Since $\vec{0} \in$ *int* K, the optimization problem

$$\max \vec{q}\vec{x} \text{ s.to } \vec{x} \in K,\ \vec{\xi}\vec{x} \ge 0$$

satisfies the Slater condition, so that $\psi_{\vec{\xi}}(\vec{q}) = \inf_{\lambda\ge 0}\max_{\vec{x}\in K}(\vec{q}\vec{x} + \lambda\vec{\xi}\vec{x}) = \inf_{\lambda\ge 0}\phi(\vec{q} + \lambda\vec{\xi})$. Consider $\vec{q} \in S^{m-1}$, where S^{m-1} is the unit sphere. Then

$$\phi(\vec{q} + \lambda\vec{\xi}) \ge \lambda\left(\phi(\vec{\xi}) + \frac{1}{\lambda}\vec{q}\nabla\phi(\vec{\xi})\right) \ge \lambda\left(\inf_{\vec{\xi}\in O}\phi(\vec{\xi}) + \frac{1}{\lambda}\inf_{\vec{q}\in S^{m-1}}\inf_{\vec{x}\in\partial K}\vec{q}\vec{x}\right),$$

where O is some neighborhood of $\vec{\xi}$. Since $\vec{0} \in intK$, we may have $\phi(\vec{\xi}) = 0$ only if $\vec{\xi} = \vec{0}$. Therefore whenever $\vec{\xi} \neq \vec{0}$, we can choose O in such a way that $\inf\limits_{\vec{\xi} \in O} \phi(\vec{\xi}) > 0$. Then if $\lambda > -2\{\inf\limits_{\vec{q} \in S^{m-1}} \inf\limits_{\vec{x} \in \partial K} \vec{q}\vec{x}\} \inf\limits_{\vec{i} \in O} \phi(\vec{\xi})$ and $\lambda > 2 \inf\limits_{\vec{\xi} \in O} \phi(\vec{\xi}) \max\limits_{\vec{q} \in S^{m-1}} \phi(\vec{q})$ we have $\phi(\vec{q}+\lambda\vec{\xi}) \geq \phi(\vec{q}+\lambda\vec{\xi})|_{\lambda=0}$. Thus, there exists a neighborhood O of the point $\vec{\xi} \neq \vec{0}$ and a positive number Λ such that for any $\vec{\xi} \in O$ we have $\psi_{\vec{\xi}}(\vec{q}) = \min\limits_{\lambda \in [0,\Lambda]} \phi(\vec{q}+\lambda\vec{\xi})$. Hence $\psi_{\vec{\xi}}(\vec{q})$ is continuous with regard to $\vec{\xi}$ and $\vec{q}$ on $O \times S^{m-1}$. Consequently, if a sequence $\vec{\xi}_n$ converges to $\vec{\xi}$ then $\psi_{\vec{\xi}_n}(\vec{q})$ converges to $\psi_{\vec{\xi}}(\vec{q})$ uniformly with regard to $\vec{q} \in S^{m-1}$. The uniform convergence of the support functions on the unit sphere is equivalent to the convergence of the respective convex compacts in terms of Hausdorf. ■

Lemma 3 *The maximization problem (17), (18)-(20), (36) satisfies the Slater condition for the constraints* $\vec{x} + \vec{z} - \vec{v} - \vec{w} \leq \vec{z}$ *and* $\vec{s}(\vec{z} - \vec{w}) \geq 0$. *Moreover, the set of solutions of this problem* $\Xi(\vec{p}, \vec{s})$ *is upper hemicontinuous with regard to* $\vec{p}, \vec{s} \in R^n_+$.

Proof. Consider $\vec{u} \in int\ R^n_+$, $\vec{u} < \vec{m}/8$ and set $y^0_{i_\nu} = f^\nu(\vec{u})/2$, $y^0_k = -u_k$ by $k \neq i_\nu$; $\vec{x}^0 = \vec{m}/16$; $\vec{z}^0 = \vec{m}/8$; $\vec{v}^0 = \vec{m}/4$; $\vec{w}^0 = \vec{m}/16$. One can easily check that the point $\langle \vec{x}^0, \vec{z}^0, \vec{v}^0, \vec{w}^0 \rangle$ belongs to the interior of the admissible set - $(\vec{p}, \vec{s})$ of the maximization problem. Also notice that there are finite distances from this point to all frontiers of the admissible set except for the one defined by the equation $\vec{s}(\vec{z} - \vec{w}) = 0$.

Let us to prove that the correspondence $\Xi(\vec{p}, \vec{s})$ has a closed graph. First, let us consider the case $\vec{s} \neq \vec{0}$. The admissible set of the problem is - $(\vec{p}, \vec{s}) = \{\vec{x}, \vec{z}, \vec{v}, \vec{w} \in R^n_+,\ \vec{y} \in Y^\nu : \vec{x} + \vec{z} - \vec{v} - \vec{w} \leq \vec{y},\ \vec{s}(\vec{z} - \vec{w}) \geq 0, \vec{v} + \vec{w} \leq 2N\vec{m}, \vec{y} \geq -3N\vec{m}\}$. Let us substitute variables using $\langle \vec{x}^0, \vec{z}^0, \vec{v}^0, \vec{w}^0 \rangle$ as a reference point $\langle \vec{x}, \vec{z}, \vec{v}, \vec{w}, \vec{y} \rangle = F(\langle \vec{x}', \vec{z}, \vec{v}', \vec{w}', \vec{y}' \rangle)$ as follows:

$$\vec{x} = \vec{x}^0 + \vec{x}';\ \vec{z} = \vec{w}^0 + \vec{w}' + \vec{z}';\ \vec{v} = \vec{v}^0 + \vec{v}';\ \vec{w} = \vec{w}^0 + \vec{w}'.$$

In terms of the new variables the admissible set is $K \cap \{\vec{x}', \vec{z}', \vec{v}', \vec{w}', \vec{y}' : \vec{s}\vec{y}' \geq 0\}$, where K is a convex compact in R^{5n} that does not depend on $\vec{p}$, $\vec{s}$ and $\vec{0} \in int$ K. The correspondence - $(\vec{p}, \vec{s})$ is represented now as - $(\vec{p}, \vec{s}) = F \circ$ - $' \circ L$, where F is the substitution function, $L : R^{2n+1} \to R^{5n}$ is a continuous correspondence $L(\vec{p}, \vec{s}) = \langle \vec{0}, \vec{s}, \vec{0}, \vec{0}, \vec{0} \rangle$ and - $'$ is the correspondence - $' : R^{5n} \to 2^K$, - $'(\vec{\omega}) = K \cap \{\vec{a} | \vec{\omega}\vec{a} \geq 0\}$. Lemma 2 implies that - $'$ is upper- and lower-hemicontinuous if $\vec{\omega} \neq \vec{0}$. F is a continuous one-to-one correspondence. L is a continuous many-to-one correspondence. L does not equal zero if $\vec{s} \neq \vec{0}$. Thus - is also upper- and lower-hemicontinuous at $\vec{s}$, $\vec{p} \in R^n$, $\vec{s} \neq \vec{0}$. Therefore $\Xi(\vec{p}, \vec{s})$ has a closed graph at this point.

Let us now consider the case $\vec{s} = \vec{0}$. Take an arbitrary sequence $\vec{s}_k \to \vec{0}$, $\vec{p}_k \to \vec{p}$, $\vec{s}_k$, $\vec{p}_k \in R^n_+$. Consider the corresponding sequences of solutions

$\langle \vec{x}_k, \vec{z}_k, \vec{v}_k, \vec{w}_k, \vec{y}_k \rangle$ which converge to some $\langle \vec{x}, \vec{y}, \vec{v}, \vec{u}, \vec{z} \rangle$. Apparently, $\langle \vec{x}, \vec{z}, \vec{v}, \vec{w}, \vec{y} \rangle$ satisfies constraints of the problem for $\vec{s} = \vec{0}, \vec{p}$. Take any other point $\langle \vec{x}', \vec{z}', \vec{v}', \vec{w}', \vec{y}' \rangle$ that meets the constraints. It is easy to check that the point $\langle \vec{x}' + \vec{z}', \vec{0}, \vec{v}' + \vec{w}', \vec{0}, \vec{y}' \rangle$ will satisfy the constraint for by $\vec{s}_k$, $\vec{p}_k$ at least for sufficiently large k. Hence

$$\vec{p}_k A^\nu \vec{x}_k + (\vec{p}_k - \vec{s}_k) A^\nu \vec{z}_k - \vec{p}_k \vec{v}_k - (\vec{p}_k - \vec{s}_k) \vec{w}_k \geq \vec{p}_k A^\nu \vec{x}' + \vec{p}_k A^\nu \vec{z}' - \vec{p}_k \vec{v}' - \vec{p}_k \vec{w}'$$

Therefore $\langle \vec{x}, \vec{y}, \vec{v}, \vec{u}, \vec{z} \rangle$ provides a value of objective function which is at least as high as one for any other point $\langle \vec{x}', \vec{y}', \vec{v}', \vec{u}', \vec{z}' \rangle$ allowed by the constraints. So, we have proved that $\Xi(\vec{p}, \vec{s})$ has a closed graph at $\vec{s} = \vec{0}$ as well. ■

Lemma 4 *Let the production sets satisfy assumptions made in the Subsection 2.1. Let $\sum_\nu \vec{y}^\nu \in R^n_+$, $\vec{y}^\nu \in Y^\nu$. If there exist $\vec{p} \in R^n_{++}$, $\vec{r}^\nu \in R^n_+$, $\vec{u}^\nu \in R^n$ and positive diagonal matrixes A^ν, $\nu = 1, \ldots, N$ such that*

$$\vec{y}^\nu \in Arg \max_{\vec{y} \in Y^\nu} \vec{r}^\nu \vec{y}^\nu \in Y^\nu, \qquad \vec{u}^\nu \geq \vec{y}^\nu, \qquad \vec{r}^\nu \in Arg \min_{\vec{r} \geq \vec{p} A^\nu} \vec{r} \vec{u}^\nu, \tag{40}$$

then $\vec{p} > \vec{0}$.

Proof. This Lemma is a generalization of the second statement in Proposition 1. Again, introduce $K = \{i : p_i = 0\}$ the set of goods with zero prices, and - $= \{\nu = 1, \ldots, N : \exists k \in K : y^\nu_{k_\nu} > 0\}$ the set of firms that produce these goods. Due to indecomposability (3) and $\sum_\nu \vec{y}^\nu \in R^n_+$, we have - $\neq \emptyset$ whenever $K \neq \emptyset$. Take an arbitrary $\nu \in$ - . The second formula in (40) implies $u^\nu_{k_\nu} > 0$. Consequently, the last part of (40) implies $r^\nu_{k_\nu} = a^\nu_{k_\nu} p^\nu_{k_\nu} = 0$. But the first condition in (40) requires $r^\nu_j = 0$ whenever $y^\nu_j \neq 0$. Since $\vec{r}^\nu \geq \vec{p} A^\nu$, we have $z^\nu_j = 0$ for all $j \notin K$, $\nu \in$ - . The rest of proof literally follows that of Proposition 1. ■

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