Model describing propagation of new technologies

1. Henkin - Polterovich's model

Levels of efficiency $n = 0, 1, \ldots$

$F_n(t)$ - the fraction of the firms which have efficiency level n or less at time $t \in [0, \infty)$. For each t the sequence $F = \{F_n(t)\}$ is a distribution function and the model describes its evolution in time. It is assumed that this evolution satisfies the following rules:

- **The efficiency can only improve in time.**
- **The firms cannot jump over levels: if a firm has a level n then it may transit to the level $n+1$ only.**
- **The speed of transition is a sum of two components: innovation and imitation components.**

$$\frac{dF_n(t)}{dt} = -(\alpha + \beta(1 - F_n(t))(F_n(t) - F_{n-1}(t))),\ \alpha, \beta > 0. \quad (1)$$

$\alpha(F_{n-1} - F_n)$ - innovation component;

$\beta(1 - F_n)(F_{n-1} - F_n)$ - imitation component.

Initial conditions:
$0 < F_n(0) \le F_{n+1}(0) < 1,\ n < N - 1,\ F_n(0) = 1,\ n \ge N$. (1')

Boundary conditions: $F_0(t) = 0$ for all $t \ge 0$. (1'')

Let us denote

$$B(t) = \prod_{k=1}^{\infty}\left(1 + \frac{\beta}{\alpha}(1 - F_k(t))\right),$$

$$F_n^*(t, A) = \left(1 + e^{-(n-c(t+\tau))}\right)^{-1},$$

where $c = \dfrac{\beta}{\ln\left(\dfrac{\alpha+\beta}{\alpha}\right)}, \quad \tau = \dfrac{\ln A}{\beta}$.

<u>Theorem</u> *(Henkin G.M., Polterovich V.M.).*

Let F_n, $1 \le n < \infty$, *be a solution of the problem (1), (1'),(1''). If we define* $A = B(0)$ *then the following estimation is valid*

$$\left|F_n(t) - F_n^*(t, A)\right| \le \lambda e^{-\gamma t}, \ 1 \le n < \infty, \ \ t \ge T_0,$$

where λ, T_0, γ *are constants depending on* $\alpha, \beta, B(0), N$.

$$\frac{dF_n(t)}{dt} = -\varphi\big(F_n(t)\big)\big(F_n(t) - F_{n-1}(t)\big)$$

<u>2.Modification of Henkin - Polterovich's model</u>

$\beta\left(F_n - F_{n+1}\right)\left(F_n - F_{n-1}\right)$ --- new imitation component

$$\frac{dF_n(t)}{dt} = -\big(\alpha + \beta\big(F_{n+1}(t) - F_n(t)\big)\big(F_n(t) - F_{n-1}(t)\big)\big). \quad (2)$$

Let $\alpha = 0$. A change of variables

$$\tau = \beta t, \ c_n(t) = F_{N+1-n}(t) - F_{N-n}(t)$$

leads to Langmuir's lattice

$$\begin{cases} \dfrac{dc_1}{dt} = c_1 c_2, \\ \dfrac{dc_n}{dt} = c_n (c_{n+1} - c_{n-1}), \quad n = 2,\ldots, N-1 \\ \dfrac{dc_N}{dt} = -c_N c_{N-1}, \\ c_n(0) = \gamma_n > 0, \qquad n = 1,\ldots, N \end{cases} \quad (4)$$

3. Investigation of Langmuir's lattice

The stable stationary solutions to (4) have the following structure

$(y_1, 0, y_2, 0, \ldots, y_k, 0)$ **if** $N = 2k$,

$(y_1, 0, y_2, 0, \ldots, y_k, 0, y_{k+1})$ **if** $N = 2k+1$.

The necessary condition for stability is

$$y_1 \geq y_2 \geq \ldots \geq y_k (\geq y_{k+1}) \geq 0.$$

J.Moser proved that if $c_1(t), \ldots, c_N(t)$ **is the solution to (4) then the eigenvalues of the Jacobi matrix**

$$L(t) = \begin{pmatrix} 0 & \sqrt{c_1(t)} & & & \\ \sqrt{c_1(t)} & 0 & \sqrt{c_2(t)} & & O \\ & \sqrt{c_2(t)} & 0 & & \\ O & & \ddots & 0 & \sqrt{c_N(t)} \\ & & & \sqrt{c_N(t)} & 0 \end{pmatrix} \quad (5)$$

don't depend on t **and are different from each other. If** N **is even then eigenvalues of the Jacobi matrix have the structure**

$$-\lambda_1 < -\lambda_2 < \ldots < -\lambda_{\frac{N+1}{2}} < \lambda_{\frac{N+1}{2}} < \ldots < \lambda_2 < \lambda_1,$$

if N is odd then the eigenvalues have the structure $-\lambda_1 < -\lambda_2 < \ldots < -\lambda_{\frac{N}{2}} < 0 < \lambda_{\frac{N}{2}} < \ldots < \lambda_2 < \lambda_1$.

The solution to (4) can be calculated analytically from the following formula proved by J.Moser

$$\cfrac{1}{z - \cfrac{c_1(t)}{z - \cfrac{c_2(t)}{z - \ddots - \cfrac{c_N(t)}{z}}}} = \frac{\sum_{n=0}^{N} \frac{m_n e^{\lambda_n^2 t}}{z - \lambda_n}}{\sum_{n=0}^{N} m_n e^{\lambda_n^2 t}} \quad . \qquad (6)$$

<u>Theorem</u> ***(Tashlitskaya Y.M., Shananin A.A.). Solutions to the Couchy problem (4) for the Lengmure's finite lattice converges as*** $t \to \infty$ ***to a fixed point, which is determined uniquely by initial data. Moreover, the following relations define the character of convergence***

$$c_{2k-1}(t) = \lambda_k^2 + O\left(e^{-\nu t}\right), c_{2k}(t) = O\left(e^{\left(\lambda_{k+1}^2 - \lambda_k^2\right)t}\right), k = 1, \ldots, n-1.$$

Here $n = \left[\frac{N}{2}\right] + 1$ ***is a number of different nonzero eigenvalues*** $\lambda_1^2 > \lambda_2^2 > \ldots > \lambda_n^2 > 0$ ***of Jacobi matrix (5), which are determined by initial data;*** $\nu = \min\left\{\lambda_k^2 - \lambda_{k+1}^2 \middle| k = 1, \ldots, n-1\right\} > 0$.

$$\frac{1}{\nu\beta} << t << \frac{1}{\alpha}.$$